Data-Driven Confidence Interval Estimation Incorporating Prior Information with an Adjustment for Skewed Data

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Bayesian credible interval (CI) estimation is a statistical procedure that has been well addressed in both the theoretical and applied literature. Parametric assumptions regarding baseline data distributions are critical for the implementation of this method. We provide a nonparametric technique for incorporating prior information into the equal-tailed (ET) and highest posterior density (HPD) CI estimators in the Bayesian manner. We propose to use a data-driven likelihood function, replacing the parametric likelihood function in order to create a distribution-free posterior. Higher order asymptotic propositions are derived to show the efficiency and consistency of the proposed method. We demonstrate that the proposed approach may correct confidence regions with respect to skewness of the data distribution. An extensive Monte Carlo (MC) study confirms the proposed method significantly outperforms the classical CI estimation in a frequentist context. A real data example related to a study of myocardial infarction illustrates the excellent applicability of the proposed technique. Supplementary material, including the R code used to implement the developed method is available online.

KEY WORDS: Credible intervals; Bayesian estimation; Nonparametric confidence interval estimation; Empirical likelihood; Equal tail confidence interval; Highest posterior density confidence interval.

1. INTRODUCTION

The Bayesian display of the upper and lower bounds of a credible set, which contains a large fraction of the posterior mass (typically 95%) related to a functional parameter, is an

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analogue of a frequentist confidence interval and commonly termed in the literature as a credible set or simply "confidence interval" (e.g., Carlin and Louis 2009:p. 35).

Because of its efficiency and natural interpretation, the Bayesian approach for confidence interval estimation is widely used in statistical practice. There is a rich statistical literature regarding the theoretical and applied aspects of the Bayesian CI estimation (e.g., Broemeling 2007; Carlin and Louis 2009; Gelman et al. 2013). To outline this technique, we assume that the dataset X consists of n independent and identically distributed observations $X = (X_1, ..., X_n)$ from density function $f(x \mid \theta)$ with an unknown parameter θ of interest. For simplicity, we suppose θ is scalar. In the Bayesian framework we define the prior distribution $\pi(\theta)$, which represents the prior information for θ mapped unto a probability space. The prior information is updated conditional on the observed data using the likelihood function $\prod_{i=1}^{n} f(X_i \mid \theta)$ via Bayes' theorem to obtain the posterior density function of θ,

$$h(\theta \mid X) = \prod_{i=1}^{n} f(X_i \mid \theta)\pi(\theta) \Big/ \int \prod_{i=1}^{n} f(X_i \mid \theta)\pi(\theta) d\theta. \qquad (1)$$

We assume that $h(\theta \mid X)$ is unimodal. The $(1-\alpha)100\%$ Bayesian CI estimate for θ can be presented as an interval $[q_L, q_U]$ such that the posterior probability $\Pr(q_L < \theta < q_U) = 1-\alpha$ at a fixed significant level α. To calculate the interval $[q_L, q_U]$, one can employ the following two strategies: (I) The CI bounds q_L and q_U are computed as roots of the equations

$$\frac{\alpha}{2} = \int_{-\infty}^{q_L} h(\theta \mid X) d\theta \quad \text{and} \quad \frac{\alpha}{2} = \int_{q_U}^{\infty} h(\theta \mid X) d\theta \quad .$$

This implies the Bayesian ET CI estimation; and (II) we can derive values of the CI bounds

q_L and q_U as roots of the equations

$$h(q_L \mid X) = h(q_U \mid X) \quad \text{and} \quad \int_{q_L}^{q_U} h(\theta \mid X) d\theta = 1 - \alpha \,.$$

This implies the Bayesian HPD CI estimation.

Method (I) is computationally simple and oftentimes used in practice. The practical implementation of Method (II) usually requires using complicated computation schemes based on Markov Chain Monte Carlo techniques (e.g., Chen and Shao 1999). It is known that Method (II) provides shorter length CI's than that of Method (I) (e.g., Carlin and Louis 2009).

In order to apply the above methods in practice, the form of density function $f(x \mid \theta)$ used in Equation (1) needs to be specified. Daniels and Hogan (2008) showed significant issues relative to verifying the parametric assumptions for various cases as to when the Bayesian CI estimation can be applied efficiently. Zhou and Reiter (2010) demonstrated that when parametric assumptions are not met exactly, the posterior estimators are generally biased. The statistical literature has displayed many examples when parametric forms of data distributions are not available and there are vital concerns relative to using the Bayesian parametric CI approach.

In this article we develop a robust nonparametric method for CI estimation in the Bayesian manner. Towards this end, we employ empirical likelihood (EL) functions to replace the corresponding unknown parametric likelihood functions in the posterior probability construction. The statistical literature has shown that the EL methodology is a very powerful data-driven method (e.g., Qin and lawless 1994; Lazar and Mykland 1998;

Owen 2001; Lazar 2003; Vexler et al. 2009). Lazar (2003) theoretically proved that EL functions can be applied for constructing nonparametric posterior distributions. In this context, EL's can provide valid posterior inference that satisfies the laws of probability in the sense that it is related to statements derived from the Bayes' rule (see for details Lazar 2003). Vexler et al. (2014) used EL functions to develop robust and efficient posterior point estimators.

Tierney and Kadane (1986) applied Laplace's method for the asymptotic evaluation of integral forms to approximate various parametric posterior quantities. We extend this method to derive higher order asymptotic approximations related to the proposed CI estimation. The derived asymptotic results show that the developed nonparametric CI estimation is more accurate than the classical CI estimation, especially in cases when data are from skewed distributions. An extensive Monte Carlo study confirms this result. We also compare the asymptotic results related to the distribution-free Bayesian CI estimation with the skewness-corrected confidence interval estimation proposed by Hall (1983). The asymptotic results we present have similar structure to those obtained by Hall (1983) when the prior information is relatively vague.

This article is organized as follows: In Section 2, we introduce a data-driven CI estimation procedure in the Bayesian manner and we derive the corresponding asymptotic properties of the proposed technique using an extension of Laplace's method. In Section 3, a Monte Carlo study is conducted to investigate the performance of the proposed CI estimation. We show that the proposed approach has very favorable properties related to the CI estimation, even when the prior function does not present correct information regarding the parameter of interest. In Section 3 we also show numerical evaluations with respect to the derived asymptotic propositions. The applicability of the proposed method is illustrated through a real world example of myocardial infarction disease in Section 4. In Section 5 we provided concluding remarks. The online supplementary material of this article consists of proofs corresponding to the theoretical results and R code to implement the proposed CI estimation.

2. METHOD

In this section we develop the EL based CI method starting with evaluations regarding a commonly used case. Without loss of generality, we begin by considering a scenario where the parameter of interest θ corresponds to the mean. This analysis provides the basic ingredients for more general statements of the problem.

Owen (1988) considered the EL technique with respect to the mean θ of a random sample $X_1, X_2, ..., X_n$. In this case the log EL function has the form

$$l(\theta) = \max_{0<p_1...p_n<1} \left\{ \sum_{i=1}^{n} \log p_i : \sum_{i=1}^{n} p_i = 1, \sum_{i=1}^{n} p_i X_i = \theta \right\},$$

where the probability weights $p_i \in (0,1)$, for $i = 1,...,n.$ Note that without further restrictions on the mean EX_1, the log EL function is

$$\max_{0<p_1,...,p_n<1} \left\{ \sum_{i=1}^{n} \log p_i : \sum_{i=1}^{n} p_i = 1 \right\} = \sum_{i=1}^{n} \log(1/n).$$

The posterior density function, $h_E(\theta \mid X)$, based on the EL function can be written as

$$h_E(\theta \mid X) = \frac{e^{lr(\theta)} \pi(\theta)}{\int_{X_{(1)}}^{X_{(n)}} e^{lr(\theta)} \pi(\theta) d\theta}, \qquad (2)$$

where $lr(\theta)=\log\left(\exp\left(l(\theta)\right)n^n\right)$ is the log EL ratio and the statistics $X_{(1)}$, $X_{(n)}$ are the minimal and maximum order statistics based on random sample $X_1, X_2, ..., X_n$. Along the Bayesian framework we use the density function $h_E(\theta \mid X)$ to construct the CI estimation through the following two strategies: the Bayesian ET and the HPD CI approaches.

2.1 Data-Driven Equal-Tailed CI Estimation

The EL based ET CI estimation requires one to find the interval $\left[q_L, q_U\right]$ that satisfies the following equations:

$$\frac{\alpha}{2}=\int_{X_{(1)}}^{q_L} h_E(\theta \mid X)d\theta \qquad \text{and} \qquad \frac{\alpha}{2}=\int_{q_U}^{X_{(n)}} h_E(\theta \mid X)d\theta . \tag{3}$$

We remark that an R function *uniroot* (R Development Core Team 2012) can be easily applied to find numerical solutions for q_L and q_U under constraints (3). For details one can see the online Supplementary material.

In order to evaluate the properties of the proposed CI estimation we present the following asymptotic results.

Proposition 1. Assume $E\mid X_1\mid^4<\infty$, and $\pi(\theta)$ is twice continuously differentiable in a neighborhood of $\bar{X}=n^{-1}\sum_{i=1}^{n} X_i$, then the estimates q_L and q_U in (3), as the sample size $n\to\infty$, satisfy the equations:

$$\frac{\alpha}{2}=\frac{\int_{X_{(1)}}^{q_L}\exp\left[-\frac{n}{2\sigma_n^2}\left(\theta-\bar{X}\right)^2\right]\pi\left(\theta\right)d\theta}{\int_{X_{(1)}}^{X_{(n)}}\exp\left[-\frac{n}{2\sigma_n^2}\left(\theta-\bar{X}\right)^2\right]\pi\left(\theta\right)d\theta}+M_n^3C_n+O_p\left(n^{-1+\varepsilon}\right), \tag{4}$$

$$1-\frac{\alpha}{2}=\frac{\int_{X_{(1)}}^{q_U}\exp\left[-\frac{n}{2\sigma_n^2}\left(\theta-\bar{X}\right)^2\right]\pi\left(\theta\right)d\theta}{\int_{X_{(1)}}^{X_{(n)}}\exp\left[-\frac{n}{2\sigma_n^2}\left(\theta-\bar{X}\right)^2\right]\pi\left(\theta\right)d\theta}+M_n^3C_n+O_p\left(n^{-1+\varepsilon}\right), \tag{5}$$

where $\overline{X} = \sum_{i=1}^{n} X_i / n$, $\sigma_n^2 = n^{-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$, $M_n^3 = n^{-1}\sum_{i=1}^{n}(X_i - \overline{X})^3$, $C_n = 2n^{-0.5}(1 + z_{1-\alpha/2}^2 / 2)\varphi\left(z_{1-\alpha/2}\right)/3\sigma_n^3$, and $\varphi(\cdot)$ denotes the standard normal density function.

This proposition simplifies the calculations of q_L and q_U. The second terms in Equations (4) and (5) involve the third moment estimate, M_n^3. Chen (1995) applied a M_n^3-type correction to improve the t-test statistic for the mean because the short right tail in the sampling distribution of t-test leads to a loss of power for tests of the population mean. This correction was shown to be a very efficient one for t-test applications based on skewed data. Proposition 1 shows that the proposed CI estimators automatically adjusts the CI estimation with respect to skewness of the data using Chen's M_n^3-type correction.

A wide range of Bayesian statistical models are based on the assumption that the prior information can be modeled via normal (Gaussian) distributions. In this case we have the following results.

Lemma 1. Assume that $\pi(\theta)$ is a normal density function with mean μ and variance σ^2. Then Equations (4) and (5) imply

$$\frac{\alpha}{2} = \Phi(u_L) + M_n^3 C_n + O_p\left(n^{-1+\varepsilon}\right), \tag{6}$$

$$1 - \frac{\alpha}{2} = \Phi(u_U) + M_n^3 C_n + O_p\left(n^{-1+\varepsilon}\right), \tag{7}$$

where $\Phi(\cdot)$ is a cumulative distribution of the standard normal random variable and

$$u_L = \left(q_L - \frac{\sigma_n^2 \mu + n\sigma^2 \overline{X}}{\sigma_n^2 + n\sigma^2}\right)\left(\frac{\sigma_n^2 \sigma^2}{\sigma_n^2 + n\sigma^2}\right)^{-0.5}, \quad u_U = \left(q_U - \frac{\sigma_n^2 \mu + n\sigma^2 \overline{X}}{\sigma_n^2 + n\sigma^2}\right)\left(\frac{\sigma_n^2 \sigma^2}{\sigma_n^2 + n\sigma^2}\right)^{-0.5}.$$

Proposition 2. Under assumptions of Proposition 1 and Lemma 1, we have

$$q_L = \frac{\sigma_n^2 \mu + n\sigma^2 \overline{X}}{\sigma_n^2 + n\sigma^2} - z_{1-\alpha/2} \sqrt{\frac{\sigma_n^2 \sigma^2}{\sigma_n^2 + n\sigma^2}} + \frac{M_n^3}{3n\sigma_n^2}\left(2 + z_{1-\alpha/2}^2\right) + o_p\left(n^{-1}\right), \quad (8)$$

$$q_U = \frac{\sigma_n^2 \mu + n\sigma^2 \overline{X}}{\sigma_n^2 + n\sigma^2} + z_{1-\alpha/2} \sqrt{\frac{\sigma_n^2 \sigma^2}{\sigma_n^2 + n\sigma^2}} + \frac{M_n^3}{3n\sigma_n^2}\left(2 + z_{1-\alpha/2}^2\right) + o_p\left(n^{-1}\right). \quad (9)$$

Regarding this result, one can remark that the first two asymptotic components of the estimator q_L and q_U are equivalent to those derived by using the Normal/Normal model in the context of the classical Bayesian CI mean estimation (e.g., Carlin and Louis 2009). In order to compare the asymptotic CI bounds (8-9) with the classical mean confidence interval estimator $\left[\overline{X} \pm z_{1-\alpha/2}\sqrt{\sigma_n^2/n}\right]$, the Equations (8) and (9) can be "informally" rewritten as

$$\left[\overline{X} \pm z_{1-\alpha/2}\sqrt{\sigma_n^2/n} + M_n^3(4 + 2z_{1-\alpha/2}^2)\big/\left(6n\sigma_n^2\right) + o_p\left(n^{-1}\right)\right]$$

when the prior hyperparameter σ^2 increases to infinity providing the case with very vague prior information. This formula is also similar to that derived by Hall (1983) to construct the skewness-corrected confidence interval estimation in the form

$$\left[\overline{X} \pm z_{1-\alpha/2}\sqrt{\sigma_n^2/n} + M_n^3(1 + 2z_{1-\alpha/2}^2)\big/\left(6n\sigma_n^2\right) + o_p\left(n^{-1}\right)\right].$$

The M_n^3-skewness correction terms involved in both the proposed CI estimator and Hall's confidence interval estimator improve the accuracy of the procedures with an adjustment for skewed data.

To prove the consistency of the proposed approach in the probabilistic manner, we present the following proposition.

Proposition 3. Under assumptions of Proposition 1, the CI bounds q_L and q_U from Equation (3) satisfy

$$\Pr(q_L < \theta < q_U) = 1 - \alpha + O_p(n^{-0.5}) .$$

2.2 Data-Driven HPD CI Estimation

In the context of the data-driven HPD CI estimation, we compute the bounds for the CI interval $[\tilde{q}_L, \tilde{q}_U]$ as the roots of the equations

$$1 - \alpha = \int_{\tilde{q}_L}^{\tilde{q}_U} h_E(\theta \mid X) d\theta \quad \text{and} \quad h_E(\tilde{q}_L \mid X) = h_E(\tilde{q}_U \mid X) . \tag{10}$$

We remark that an R function *optim* (R Development Core Team 2012) can be easily used to find numerical solutions for $\tilde{q}_L$ and $\tilde{q}_U$ under the constraints at (10). For details one can see the online Supplementary material. In a similar manner to Section 2.1, we present the following asymptotic result for the HPD CI estimation.

Proposition 4. Under the assumptions of Proposition 1, the estimates $\tilde{q}_L$ and $\tilde{q}_U$ in (10) satisfy the equations

$$\begin{aligned}
&\exp\left[-n\left(\tilde{q}_L - \bar{X}\right)^2 \Big/ 2\sigma_n^2 + nM_n^3\left(\tilde{q}_L - \bar{X}\right)^3 \Big/ 3\left(\sigma_n^2\right)^3 + O_p\left(n\right)\left(\tilde{q}_L - \bar{X}\right)^4\right]\pi\left(\tilde{q}_L\right) \\
&= \exp\left[-n\left(\tilde{q}_U - \bar{X}\right)^2 \Big/ 2\sigma_n^2 + nM_n^3\left(\tilde{q}_U - \bar{X}\right)^3 \Big/ 3\left(\sigma_n^2\right)^3 + O_p\left(n\right)\left(\tilde{q}_U - \bar{X}\right)^4\right]\pi\left(\tilde{q}_U\right),
\end{aligned}$$

$$\begin{aligned}
&\int_{\tilde{q}_L}^{\tilde{q}_U} \exp\left[-\frac{n}{2\sigma_n^2}\left(\theta - \bar{X}\right)^2\right]\pi\left(\theta\right)d\theta \Big/ \int_{X_{(1)}}^{X_{(n)}} \exp\left[-\frac{n}{2\sigma_n^2}\left(\theta - \bar{X}\right)^2\right]\pi\left(\theta\right)d\theta \\
&= 1 - \alpha + O_p\left(n^{-1+\varepsilon}\right).
\end{aligned}$$

The result of Proposition 4 can be simplified given that $\pi(\theta)$ is a normal density function with mean μ and variance σ^2. Lemma 2 and Proposition 5 represent the results.

Lemma 2. Under assumptions of Lemma 1, $\tilde{q}_L$ and $\tilde{q}_U$ can be approximated as the roots of the equations

$$-n\left(\tilde{q}_L-\bar{X}\right)^2\Big/2\sigma_n^2+nM_n^3\left(\tilde{q}_L-\bar{X}\right)^3\Big/3\left(\sigma_n^2\right)^3-\left(\tilde{q}_L-\mu\right)^2\Big/2\sigma_n^2+O_p\left(n\right)\left(\tilde{q}_L-\bar{X}\right)^4$$
$$=-n\left(\tilde{q}_U-\bar{X}\right)^2\Big/2\sigma_n^2+nM_n^3\left(\tilde{q}_U-\bar{X}\right)^3\Big/3\left(\sigma_n^2\right)^3-\left(\tilde{q}_U-\mu\right)^2\Big/2\sigma_n^2+O_p\left(n\right)\left(\tilde{q}_U-\bar{X}\right)^4,$$

$$\Phi\left(u_U\right)-\Phi\left(u_L\right)=1-\alpha+O_p\left(n^{-1+\varepsilon}\right).$$

Proposition 5. Under the assumptions of Proposition 1 and Lemma 1, the asymptotic expansions for the bounds of the HPD CI, $\left[\tilde{q}_L,\tilde{q}_U\right]$, are

$$\tilde{q}_L=\bar{X}-z_{1-\alpha/2}\sqrt{\frac{\sigma_n^2\sigma^2}{\sigma_n^2+n\sigma^2}}+\left[\frac{M_n^3z_{1-\alpha/2}^2}{3n\sigma_n^2}-\frac{\sigma_n^2}{n\sigma^2}\left(\bar{X}-\mu\right)\right]+o_p\left(n^{-1}\right), \qquad (11)$$

$$\tilde{q}_U=\bar{X}+z_{1-\alpha/2}\sqrt{\frac{\sigma_n^2\sigma^2}{\sigma_n^2+n\sigma^2}}+\left[\frac{M_n^3z_{1-\alpha/2}^2}{3n\sigma_n^2}-\frac{\sigma_n^2}{n\sigma^2}\left(\bar{X}-\mu\right)\right]+o_p\left(n^{-1}\right). \qquad (12).$$

For the consistency of the proposed HPD approach we present the next Proposition.

Proposition 6. Under the assumptions of Proposition 3, the CI bounds $\tilde{q}_L$ and $\tilde{q}_U$ in Equation (10) satisfy

$$\Pr(\tilde{q}_L<\theta<\tilde{q}_U)=1-\alpha+O_p(n^{-0.5}).$$

2.3 The General Case of the Data-Driven CI Estimation

In order to extend the results developed in the previous sections, we define the log EL function as

$$l_G(\theta)=\max_{0<p_1,\ldots,p_n<1}\left\{\sum_{i=1}^{n}\log p_i:\sum_{i=1}^{n}p_i=1,\sum_{i=1}^{n}p_iG(X_i,\theta)=0\right\},$$

where the probability weights $p_i\in(0,1)$, for $i=1,\ldots,n.$ We assume for simplicity that $\partial G(u,\theta)/\partial\theta>0$ or $\partial G(u,\theta)/\partial\theta<0$, for all u. Define θ_M as a solution of the equation $n^{-1}\sum_{i=1}^{n}G(X_i,\theta)=0$. In the case of $G(X_i,\theta)=X_i-\theta$, the parameter θ is the mean of data that is considered in Sections 2.1 and 2.2. We construct the nonparametric posterior density function as

$$h_{EG}(\theta \mid X) = \frac{e^{lr_G(\theta)}\pi(\theta)}{\int_{X_{(1)}}^{X_{(n)}} e^{lr_G(\theta)}\pi(\theta)d\theta},$$

where $lr_G(\theta) = \log\left(\exp\left(l_G(\theta)\right)n^n\right)$ is the log EL ratio. The posterior density function $h_{EG}(\theta \mid X)$ can be applied to denote the data-driven CI estimation via two strategies, the Bayesian ET and HPD CI estimations. Toward this end one can use Equations (3) and (10) by applying the function $h_{EG}(\theta \mid X)$ instead of $h_E(\theta \mid X)$.

In a similar manner to the CI evaluations presented in Sections 2.1 and 2.2 regarding the data-driven ET CI estimation with bounds $[Q_L, Q_U]$, we have the following asymptotic propositions.

Proposition 7. Assume $E \mid G(X_1,\theta) \mid^4 < \infty$, and $\pi(\theta)$ is twice continuously differentiable in a neighborhood of θ_M, then the lower and upper CI bounds Q_L and Q_U satisfy the equations

$$\frac{\alpha}{2} = \frac{\int_{X_{(1)}}^{Q_L} \exp\left[-\frac{n}{2\sigma_{Gn}^2}\left(\theta-\theta_M\right)^2\right]\pi\left(\theta\right)d\theta}{\int_{X_{(1)}}^{X_{(n)}} \exp\left[-\frac{n}{2\sigma_{Gn}^2}\left(\theta-\theta_M\right)^2\right]\pi\left(\theta\right)d\theta} + M_{Gn}^3 C_{Gn} + O_p\left(n^{-1+\varepsilon}\right),$$

$$1\text{-}\frac{\alpha}{2} = \frac{\int_{X_{(1)}}^{Q_U} \exp\left[-\frac{n}{2\sigma_{Gn}^2}\left(\theta-\theta_M\right)^2\right]\pi\left(\theta\right)d\theta}{\int_{X_{(1)}}^{X_{(n)}} \exp\left[-\frac{n}{2\sigma_{Gn}^2}\left(\theta-\theta_M\right)^2\right]\pi\left(\theta\right)d\theta} + M_{Gn}^3 C_{Gn} + O_p\left(n^{-1+\varepsilon}\right),$$

where $\sigma_{Gn}^2 = n^{-1}\sum_{i=1}^n G(X_i,\theta)^2$, $M_{Gn}^3 = n^{-1}\sum_{i=1}^n G(X_i,\theta)^3$ and

$$C_{Gn} = 2n^{-0.5}\left(1+z_{1-\alpha/2}^2/2\right)\varphi\left(z_{1-\alpha/2}\right)/3\sigma_{Gn}^3.$$

The next proposition provides the asymptotic evaluations of the HPD CI estimation with the bounds $\left[\widehat{Q}_L, \widehat{Q}_U\right]$.

Proposition 8. Under the assumptions of Proposition 7, the lower and upper HPD CI bounds, $\hat{Q}_L$ and $\hat{Q}_U$, asymptotically satisfy the equations

$$\exp\left[-n\left(\hat{Q}_L-\overline{X}\right)^2\Big/2\sigma_{Gn}^2+nM_n^3\left(\hat{Q}_L-\overline{X}\right)^3\Big/3\left(\sigma_{Gn}^2\right)^3+O_p\left(n\right)\left(\hat{Q}_L-\overline{X}\right)^4\right]\pi(\hat{Q}_L)$$
$$=\exp\left[-n\left(\hat{Q}_U-\overline{X}\right)^2\Big/2\sigma_{Gn}^2+nM_n^3\left(\hat{Q}_U-\overline{X}\right)^3\Big/3\left(\sigma_{Gn}^2\right)^3+O_p\left(n\right)\left(\hat{Q}_U-\overline{X}\right)^4\right]\pi(\hat{Q}_U),$$

$$\int_{\hat{Q}_L}^{\hat{Q}_U}\exp\left[-\frac{n}{2\sigma_{Gn}^2}\left(\theta-\theta_M\right)^2\right]\pi\left(\theta\right)d\theta\Big/\int_{X_{(1)}}^{X_{(n)}}\exp\left[-\frac{n}{2\sigma_{Gn}^2}\left(\theta-\theta_M\right)^2\right]\pi\left(\theta\right)d\theta$$
$$=1+O_p(n^{-1+\varepsilon}).$$

We note that one can easily derive asymptotic expression for the CI bounds in general case by directly using the proof strategies of Propositions 2 and 5 when the prior distribution is in Gaussian form. The following proposition confirms that the proposed nonparametric procedures are consistent.

Proposition 9. Under assumptions of Proposition 7, we have

$$\Pr(Q_L<\theta<Q_U)=1-\alpha+O_p(n^{-0.5}) \quad \text{and} \quad \Pr(\hat{Q}_L<\theta<\hat{Q}_U)=1-\alpha+O_p(n^{-0.5}).$$

Remark 1. We defined the log EL function, $l_G(\theta)$ in a manner that has been extensively dealt with in the EL literature (e.g., Owen, 1988, 2001; Vexler at al. 2014). In this case, the constraint $\sum_{i=1}^n G(X_i,\theta)=0$ reflects empirically the functional meaning of the parameter θ in the form $E\left(G\left(X_1,\theta\right)\right)=0$, where G is known. One can consider scenarios when the function G is unknown, for example, in the context of quantile estimation (e.g., Chen and Hall 1993). In a subsequent paper, we plan to address this problem. Further studies are needed to evaluate the Bayesian type EL CI approach in this framework.

3. MONTE CARLO STUDY

In this section, we carry out an extensive MC study in order to evaluate the behavior of the proposed CI estimation with fixed significance level set at 5%. Limpert et al. (2001) showed that many measurements of markers related to health and social science have skewed distributions. Toward this end, we focus on simulating data for this numerical study following the distribution function $\Pr(X_1 < t) = \int_0^{t+\exp(\sigma_1^2/2)} f(x)dx$, where $f(x)$ is a $LogNorm(0,\sigma_1{}^2)$ density distribution with σ_1 = 1, 1.5, 2 and $\theta = EX_1 = 0$. It is known that for normally distributed data EL methods demonstrate good properties, whereas EL type procedures based on lognormal-type distributed data can lead to unstable results (e.g., Vexler at el. 2009). In these Monte Carlo evaluations we also generated data from a $N(0,1)$ distribution. We consider the following prior distributions in our simulation study: $\pi(\theta)$ = $(2\pi\sigma_\pi^2)^{-0.5}\exp\left(-(\theta-d)^2/(2\sigma_\pi^2)\right)$ with $d = 0,1$ and σ_π = 0.25, 0.5, 1. These priors reflect different scenarios depicting our "relative confidence" with respect to the prior information pertaining to the unknown parameter $\theta = EX$. At each baseline distribution and prior density function, we generated 5000 samples of size n = 7, 15, 25, 50, 100. The 95% classical nonparametric CI is $\left[\bar{X} \pm 1.96\sqrt{\sigma_n^2/n}\right]$, where σ_n^2 is the sample variance. The proposed method uses the EL functions instead of the parametric joint density functions in Bayesian CI estimation. In several scenarios, Bayesian credible sets can demonstrate poor frequentist properties (e.g., Szabó et al. 2015). Proposition 3 ensures that the coverage probability of the new data-driven CI estimation is controlled asymptotically in the frequentist context. We evaluate the frequentist coverage based on finite samples. The criteria for comparison are the MC coverage probability (CP) and the MC average length of the CI (LG). Table I demonstrates the MC results.

For the cases of lognormally distributed data with priors $(2\pi\sigma_{\pi}^{2})^{-0.5}\exp\left(-(\theta-d)^{2}/(2\sigma_{\pi}^{2})\right)$ that provide correct information regarding θ when $d=0$, the CPs of the proposed CI estimations (both the ET and HPD CI estimators) are almost uniformly closer to the expected 95% level than those of the classical CI estimation. The LGs of the proposed method are shorter in most cases. When the skewness of the lognormal distribution increases, the above conclusions are magnified. When using the misspecified prior, $N(1,1)$ $(d=1)$, the proposed method maintains similar CPs with a little increase in LGs as comparing to the classical method. For the cases of baseline data from normal distributions with the correctly specified priors, the performance of the proposed methods are comparable to the classical method. In these cases the classical CI method is a product of the parametric maximum likelihood technique and then can be expected to be very efficient.

In the Supplementary Material we compare the proposed nonparametric approach with the following methods: (1) the inverse Edgeworth expansion based method proposed by Hall (1983); (2) the parametric Bayesian CI estimation; (3) a frequentist method for improved CI estimation of the log-normal mean; and (4) the classical EL confidence interval estimation (Owen 2001). In the considered MC scenarios, the data-driven CI estimation outperforms Hall's approach. Perhaps, this event is a result of the fact that in order to obtain the confidence intervals using Hall's method it is required to estimate several unknown parameters within the corresponding asymptotic approximations. The estimators of the parameters can be very biased when skewed data are used. The proposed CI approach demonstrates better CPs and LPs than those provided by the classical EL method. In the context of the comparisons with (2) and (3), the new distribution-free CI estimation shows results that are very close to outputs of the parametrical methods.

Table 1. The Monte Carlo coverage probabilities (CP) and average lengths (LG) for the CI estimation of the mean. The Z indicates the results of the classical CI estimation.

	$X_i \sim \exp(\zeta_i)-\exp(0.5)$, $\zeta_i \sim N(0,1), i=1,\ldots,n.$						$X_i \sim \exp(\zeta_i)-\exp(0.5)$, $\zeta_i \sim N(0,1), i=1,\ldots,n.$					
	Prior: $\pi \sim N(0,1)$						Prior: $\pi \sim N(0,0.5)$					
	Z		ET		HPD		Z		ET		HPD	
N	CP	LG	CP	LG	CP	LG	CP	LG	CP	LG	CP	LG
7	75.7%	2.33	76.5%	1.52	75.8%	1.50	78.1%	2.54	80.1%	1.19	79.8%	1.19
15	83.5%	1.88	86.6%	1.45	86.3%	1.43	82.9%	1.81	90.0%	1.14	89.7%	1.14
25	86.8%	1.51	89.8%	1.31	89.5%	1.28	86.6%	1.46	92.5%	1.06	91.9%	1.05
50	89.9%	1.10	92.1%	1.07	92.3%	1.05	89.9%	1.10	93.8%	0.91	93.7%	0.90
100	91.6%	0.78	92.9%	0.81	93.1%	0.80	91%	0.80	94.1%	0.74	94.7%	0.73
	Prior: $\pi \sim N(0,0.25)$						Prior: $\pi \sim N(1,1)$					
	Z		ET		HPD		Z		ET		HPD	
N	CP	LG	CP	LG	CP	LG	CP	LG	CP	LG	CP2	LG
7	78.6%	2.32	83.6%	0.79	84.3%	0.79	77.1%	2.38	75%	1.65	76.8%	1.65
15	82.9%	1.76	93.1	0.79	93.0	0.79	82.9%	1.80	84.4%	1.59	85.9%	1.57
25	85.6%	1.45	95.8%	0.76	95.4%	0.76	87.7%	1.50	86.6%	1.46	88.3%	1.43
50	89.9%	1.10	96.97%	0.70	96.8%	0.69	89.9%	1.10	90.0%	1.17	90.5%	1.14
100	91.8%	0.81	96.96%	0.60	96.7%	0.60	91.8%	0.81	90.1%	0.89	91.5%	0.87
	$X_i \sim \exp(\zeta_i)-\exp(2)$, $\zeta_i \sim N(0,2), i=1,\ldots,n.$						$X_i \sim \exp(\zeta_i)-\exp(1.5^2/2)$, $\zeta_i \sim N(0,1.5), i=1,\ldots,n.$					
	Prior: $\pi \sim N(0,1)$						Prior: $\pi \sim N(0,1)$					
	Z		ET		HPD		Z		ET		HPD	
n	CP	LG	CP	LG	CP	LG	CP	LG	CP	LG	CP	LG
7	50.8%	20.5	63.9%	3.04	64.1%	3.04	64.2%	7.34	70.7%	2.46	69.9%	2.45
15	56.3%	18.6	78.8%	3.46	78.7%	3.46	71.8%	5.60	81.0%	2.51	80.3%	2.50
25	66.3%	17.51	88.3%	3.60	88.3%	3.60	75.8%	4.57	86.8%	2.45	85.3%	2.43
50	69.6%	12.79	92.6%	3.60	92.5%	3.59	81.7%	3.54	90.9%	2.28	89.7%	2.26
100	74.8%	10.38	95.7%	3.51	95.7%	3.50	84.1%	3.05	93.8%	2.03	92.5%	2.00
	$X_1,\ldots,X_n \sim N(0,1)$,						$X_1,\ldots,X_n \sim N(0,1)$,					
	Prior: $\pi \sim N(0,0.5)$						Prior: $\pi \sim N(0,1)$					
	Z		ET		HPD		Z		ET		HPD	

n	CP	LG	CP	LG	CP	LG	CP	LG	CP	LG	CP	LG
7	89.8%	1.42	88.5%	1.03	89.1%	1.02	91.3%	1.44	88.4%	1.20	88.6%	1.19
15	92.5%	0.99	94.6%	0.87	94.7%	0.86	91.2%	0.99	90.9%	0.95	91.1%	0.94
25	94.2%	0.78	95.7%	0.73	95.8%	0.73	93.4%	0.78	93.9%	0.77	94.0%	0.77
50	94.1%	0.55	95.4%	0.54	95.4%	0.54	94.1%	0.5	94.6%	0.56	94.5%	0.56
100	95.3%	0.39	95.8%	0.39	95.9%	0.39	94.9%	0.39	95.4$	0.39	95.4%	0.39

To evaluate the accuracy of Propositions 2 and 5, 5000 samples of sizes n = 50, 200, 300, 400 and 500 were generated from both the normal distribution $N(0,1)$ and the exponential distribution $Exp(1)$. We used the prior density function $\pi(\theta) = (2\pi)^{-0.5}\exp\left(-(\theta)^2/2\right)$. For each MC generation, we calculated values of the quantiles $\Delta_1 = n\left|q_L - q_{AL}\right|$ and $\Delta_2 = n\left|\tilde{q}_L - \tilde{q}_{AL}\right|$, where $q_{AL} = n\bar{X}/(\sigma_n^2+n) - 1.96\sigma_n/(\sigma_n^2+n) + 1.95M_n^3/n\sigma_n^2$ and $\tilde{q}_{AL} = \bar{X} - 1.96\sigma_n/(\sigma_n^2+n)^{-0.5} + (1.28M_n^3/n\sigma_n^2 - \bar{X}\sigma_n^2/n)$ such that q_{AL} and $\tilde{q}_{AL}$ represent the asymptotic ET and HPD CI bounds defined in Equations (8) and (11), respectively. We plot the Monte Carlo average values of Δ_1 and Δ_2 against the sample sizes in Figure 1. The MC results showed the asymptotic propositions provide accurate approximations.

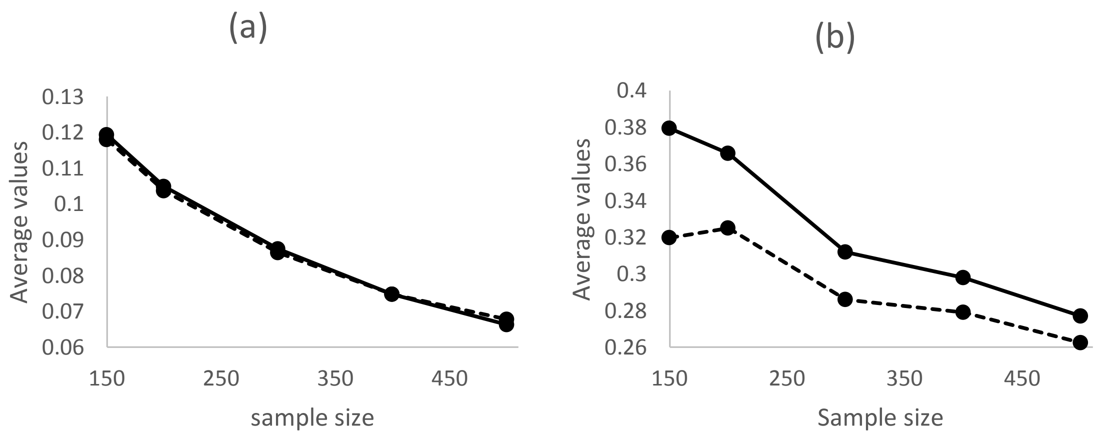

Figure 1. The numerical evaluations of the results of Propositions 2 and 5. The plot of the average values of Δ_1 and Δ_2 (curves "–" and "- -" respectively) against sample

sizes, the x-axis. Panel (a) and Panel (b) represents the results based on X~ $N(0,1)$ and X~ $Exp(1)$, respectively.

4. DATA EXAMPLE

In this section, a real-life data example is presented in order to illustrate the applicability of the proposed method. The example is based on a sample from a study that evaluates biomarkers related to the myocardial infarction (MI). The study was focused on the residents of Erie and Niagara counties, 35-79 years of age. The New York State department of Motor Vehicles drivers' license rolls was used as the sampling frame for adults between the age of 35 and 65 years, while the elderly sample (age 65-79) was randomly chosen from the Health Care Financing Administration database. We consider the biomarker "glucose" that is often used as a discriminant factor between individuals with and without MI disease (e.g., Schisterman et al., 2001). A total of 386 measurements of glucose biomarker were evaluated by the study. Half of them were collected on cases who survived on MI and the other half on controls who had no previous MI. In order to implement the proposed method, results of the population-based study described in Schisterman et al. (2001) were employed to provide the prior information of the parameter, the population mean of the glucose level. Based on the research of Schisterman et al. (2001), it is reasonable to assume that $N(105.04,\ 33.38^2)$ and $N(161.85,\ 68.04^2)$ are the prior distributions for the mean of glucose in the control and the case groups, respectively. Table 2 presents the results of the proposed 95% CI estimation of the mean of the glucose biomarker.

Based on the Shapiro-Wilk test of normality we reject the normality assumption for the glucose data for both the case and control groups (p-values < 0.05) due to the right-skewed distributions. The non-overlapping CI's provided by the proposed method for the case and control groups suggest there is a significant difference in means of glucose levels between the two groups. By contrast, the 95% classical CI's for the case and control groups overlap and do not provide this conclusion.

Table 2. The 95% CI estimators of the mean of the glucose biomarker.

	Equal-tailed CI	HPD CI	Classical CI
Case group	[106.17, 116.85]	[105.87, 116.46]	[105.26, 115.55]
Control group	[99.78, 105.98]	[99.61, 105.76]	[99.28, 105.28]

5. CONCLUDING REMARKS

The nonparametric technique for incorporating prior information into the CI estimation in the Bayesian manner was developed. The asymptotic propositions showed that the proposed method can improve the CI estimation with an adjustment for skewed data. The Monte Carlo study confirmed that the proposed CI estimation is more accurate relative to the coverage probability aspect than that of the classical CI estimation and has shorter length of CI estimation. To demonstrate the applicability of the proposed method we applied our method to the study of a glucose biomarker for myocardial infarction.

SUPPLEMENTAR MATERIALS

Appendix: Detailed proofs of the propositions presented in the article.

R Code: R Code to implement the developed CI estimators in the article.

Additional Monte Carlo results.

ACKNOWLEDGMENT

We are grateful to the Editor, Associate Editor and reviewers for their helpful comments that led to a substantial improvement in this paper.

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On-line Supplement to "Data-Driven Confidence Interval Estimation Incorporating Prior Information with an Adjustment for Skewed Data"

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Abstract: This on-line supplement to "Data-Driven Confidence Interval Estimation Incorporating Prior Information with an Adjustment for Skewed Data" contains an appendix of technical proofs of Propositions and Lemmas and R code to implement the developed methods proposed in the article. In this Supplement Material we also show additional results of the Monte Carlo evaluations of the proposed method.

APPENDIX

Proof of Proposition 1

The proof of Proposition 1 is based on the fact that function $lr(\theta)$ is highly peaked about its maximum $\overline{X}=\sum_{i=1}^{n} X_i \big/ n$. We will use the fact that $lr(\theta)$ can be well approximated by the function $-n(\theta-\overline{X})^2/2\sigma_n^2$, when values of θ are close to $\overline{X}$. Toward this end, we first show that

$$\int_{X_{(1)}}^{X_{(n)}} e^{lr(\theta)}\pi(\theta)d\theta=\int_{\overline{X}-\varphi_n n^{-1/2}}^{\overline{X}+\varphi_n n^{-1/2}} e^{lr(\theta)}\pi(\theta)d\theta+O\left(\exp\left(-c\varphi_n^2\right)\right),$$

where $c>0$ is a constant, a positive sequence $\varphi_n=O\left(n^{\varepsilon}\right)\to\infty$, for some $\varepsilon>0$, $\varphi_n n^{-0.5}\to 0$ and $\int_{\overline{X}-\varphi_n n^{-1/2}}^{\overline{X}+\varphi_n n^{-1/2}} e^{lr(\theta)}\pi(\theta)d\theta=O\left(n^{-1/2}\right)$, as $n\to\infty$. This approximation allows us to analyze the numerator and denominator in (3). Denote the log EL function

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$l(\theta)=\sum_{i=1}^{n}\log p_i$, where p_i's maximize the Lagrangian

$$\Delta=\sum_{i=1}^{n}\log(p_i)+\lambda_1(1-\sum_{i=1}^{n}p_i)+\lambda_2(\theta-\sum_{i=1}^{n}p_iX_i),$$

with λ_1, λ_2 that are the Lagrange multipliers. One can show that $p_i=\left(n+\lambda\left(X_i-\theta\right)\right)^{-1}$, where λ is a root of the equation $\sum_{i=1}^{n}\left(X_i-\theta\right)/\left(n+\lambda\left(X_i-\theta\right)\right)=0$.

By virtue of Lemma 10.2.1 in Vexler et al. (2014a), when $\theta<\bar{X}$, $lr(\theta)$ is increasing; when $\theta>\bar{X}$, $lr(\theta)$ is decreasing. This implies that the log empirical likelihood ratio function $lr(\theta)$ defined in (2) has the maximum at $\theta=\bar{X}$. Denote $a=\bar{X}-\varphi_n n^{-0.5}$ and $b=\bar{X}+\varphi_n n^{-0.5}$ where $\varphi_n=n^{1/6-\beta}$ and $\beta\in\left(0,1/6\right)$. Then it turns out that

$$\int_{X_{(1)}}^{X_{(n)}}e^{lr(\theta)}\pi\left(\theta\right)d\theta=\int_{X_{(1)}}^{a}e^{lr(\theta)}\pi\left(\theta\right)d\theta+\int_{a}^{b}e^{lr(\theta)}\pi\left(\theta\right)d\theta+\int_{b}^{X_{(n)}}e^{lr(\theta)}\pi\left(\theta\right)d\theta,$$

and $\int_{X_{(1)}}^{q_L}e^{lr(\theta)}\pi\left(\theta\right)d\theta=\int_{X_{(1)}}^{a}e^{lr(\theta)}\pi\left(\theta\right)d\theta+\int_{a}^{q_L}e^{lr(\theta)}\pi\left(\theta\right)d\theta$.

By the virtue of the above considerations we can bound the remainder term

$$\int_{X_{(1)}}^{a}e^{lr(\theta)}\pi(\theta)d\theta\le e^{lr(a)}\int_{X_{(1)}}^{X_{(n)}}\pi(\theta)d\theta\le e^{lr(a)}.$$

In order to evaluate $lr(a)$, we define the function

$$L(\lambda)=\sum_{i=1}^{n}(X_i-\theta)/(n+\lambda(X_i-\theta)). \tag{1.1}$$

We rewrite (1.1) at $\theta=a$ such that

$$\begin{aligned}L(\lambda)&=\sum_{i=1}^{n}(X_i-\bar{X}+\varphi_n n^{-0.5})/(n+\lambda(X_i-\bar{X}+\varphi_n n^{-0.5}))\\&=\frac{1}{n}\sum_{i=1}^{n}\frac{(X_i-\bar{X}+\varphi_n n^{-0.5})\left[\left(1+\lambda n^{-1}(X_i-\bar{X}+\varphi_n n^{-0.5})\right)-\lambda n^{-1}(X_i-\bar{X}+\varphi_n n^{-0.5})\right]}{1+\lambda n^{-1}(X_i-\bar{X}+\varphi_n n^{-0.5})}\\&=\frac{1}{n}\left[\sum_{i=1}^{n}(X_i-\bar{X}+\varphi_n n^{-0.5})-\lambda n^{-1}\sum_{i=1}^{n}\frac{(X_i-\bar{X}+\varphi_n n^{-0.5})^2}{1+\lambda n^{-1}(X_i-\bar{X}+\varphi_n n^{-0.5})}\right].\end{aligned} \tag{1.2}$$

Defining $\lambda_c=n^{2/3}\tau_n^{-1}$, where $\tau_n=n^{\gamma}$, $0<\gamma<\beta<1/6$, and plugging it into (1.2) we have

$$\sqrt{n}L(\lambda_c)=\varphi_n-\sqrt{n}\frac{n^{2/3-1}}{\tau_n}\frac{1}{n}\sum_{i=1}^{n}\frac{(X_i-\bar{X}+\varphi_n n^{-0.5})^2}{1+n^{-1/3}\tau_n^{-1}(X_i-\bar{X}+\varphi_n n^{-0.5})},$$

Since $\frac{X_i - \bar{X}}{n^{1/3}\tau_n} = O_p(1)$ (e.g., Owen 1988), we have

$$\sqrt{n}L(\lambda_c) = \varphi_n - \frac{n^{1/6}}{\tau_n}\frac{1}{n}\sum_{i=1}^{n}\frac{(X_i - \bar{X} + \varphi_n n^{-0.5})^2}{1 + O_p(1)}.$$

Thus $\sqrt{n}L(\lambda_c) \to -\infty$, as $n \to \infty$. In a similar manner, $\sqrt{n}L(-\lambda_c) \to \infty$, as $n \to \infty$. And then the solution, λ_0, of equation $\sqrt{n}L(\lambda_0) = 0$ belongs to the interval $[-\lambda_c, \lambda_c]$, i.e. $\lambda_0 = O_p\left(n^{2/3}\tau_n^{-1}\right)$.

Let us now derive the approximate value of λ_0 as $n \to \infty$. Since $L(\lambda_0) = 0$,

$$\sum_{i=1}^{n}\frac{(X_i - \bar{X} + \varphi_n n^{-0.5})}{1 + \lambda_0 n^{-1}(X_i - \bar{X} + \varphi_n n^{-0.5})} = 0. \qquad (1.3)$$

Applying a Taylor series expansion to (1.3) considering $\lambda_0 n^{-1}(X_i - \bar{X} + \varphi_n n^{-0.5})$ around zero we obtain

$$\sum_{i=1}^{n}\left(X_i - \bar{X} + \varphi_n n^{-0.5}\right)\left[1 - \lambda_0 n^{-1}\left(X_i - \bar{X} + \varphi_n n^{-0.5}\right) + \frac{\lambda_0{}^2 n^{-2}\left(X_i - \bar{X} + \varphi_n n^{-0.5}\right)^2}{\left(1 + \omega_i\right)^3}\right] = 0. \qquad (1.4)$$

where $0 < \omega_i < \lambda_0 n^{-1}(X_i - \bar{X} + \varphi_n n^{-0.5})$. Since $\lambda_0 = O_p\left(n^{2/3}\tau_n^{-1}\right)$, we can rewrite (1.4) in the form

$$\sum_{i=1}^{n}\left(X_i - \bar{X} + \frac{\varphi_n}{n^{1/2}}\right) - \frac{\lambda}{n}\sum_{i=1}^{n}\left(X_i - \bar{X} + \frac{\varphi_n}{n^{1/2}}\right)^2 + \frac{O(n^{1/3})}{\tau_n^2}\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X} + \frac{\varphi_n}{n^{1/2}}\right)^3 = 0. \qquad (1.5)$$

Then solving (1.5) gives the approximate solution by

$$\lambda_0 = \frac{\varphi_n n^{1/2}}{n^{-1}\sum_{i=1}^{n}(X_i - \bar{X} + \varphi_n n^{-1/2})^2} + \frac{O(n^{1/3})}{\tau_n^2}. \qquad (1.6)$$

Applying a Taylor series expansion to $lr(a)$ considering $\lambda_0 n^{-1}(X_i - \bar{X} + \varphi_n n^{-0.5})$ around zero yields

$$lr(a) = -\sum_{i=1}^{n}\log\left[1 + \frac{\lambda_0}{n}(X_i - \bar{X} + \varphi_n n^{-0.5})\right]$$

$$= -\sum_{i=1}^{n}\frac{\lambda_0}{n}(X_i - \bar{X} + \varphi_n n^{-0.5}) + \frac{1}{2}\sum_{i=1}^{n}\frac{\lambda_0{}^2}{n^2}(X_i - \bar{X} + \varphi_n n^{-0.5})^2 - \frac{1}{3}\sum_{i=1}^{n}\frac{\lambda_0{}^3}{n^3}\frac{(X_i - \bar{X} + \varphi_n n^{-0.5})^3}{(1 + \omega_i^*)^3},$$

where $0 < \omega_i^* < \lambda_0 n^{-1}(X_i - \bar{X} + \varphi_n n^{-0.5})$. By virtue of (1.6) and the fact that

$\lambda_0 = O(n^{2/3}/\tau_n)$ we have

$$lr(a) = -\frac{\lambda_0}{n}\varphi_n n^{1/2} + \frac{1}{2}\sum_{i=1}^{n}\frac{\lambda_0^{\ 2}}{n^2}(X_i - \bar{X} + \varphi_n n^{-0.5})^2 - O(n^{-3\gamma})$$

$$= \frac{-\varphi_n^{\ 2} n}{nn^{-1}\sum_{i=1}^{n}(X_i - \bar{X} + \varphi_n n^{-0.5})^2} - \frac{O(n^{4/3})}{\tau_n^2 n^2}\varphi_n n^{1/2} + \frac{1}{2}\left[\frac{\varphi_n^{\ 2} n}{[n^{-1}\sum_{i=1}^{n}(X_i - \bar{X} + \varphi_n n^{-0.5})^2]^2}\right.$$

$$\left. + 2\frac{O(n^{4/3})}{\tau_n^2 n}\frac{\varphi_n^{\ 2} n^{1/2}}{n^{-1}\sum_{i=1}^{n}(X_i - \bar{X} + \varphi_n n^{-0.5})^2} + \frac{O(n^{8/3})}{\tau_n^4 n^2}\right]\frac{1}{n^2}\sum_{i=1}^{n}(X_i - \bar{X} + \varphi_n n^{-0.5})^2 - O(n^{-3\gamma})$$

$$= -\frac{1}{2}\frac{\varphi_n^{\ 2}}{n^{-1}\sum_{i=1}^{n}(X_i - \bar{X} + \varphi_n n^{-0.5})^2} - O(n^{-3\gamma}) \to -\infty \text{ , as } n \to \infty,$$

where $\varphi_n^{\ 2} = n^{1/3-2\beta} \to \infty$ and $0 < \gamma < \beta < 1/6$. Thus we conclude that

$$\int_{X_{(1)}}^{a} \exp\left(lr(\theta)\right)\pi(\theta)d\theta \le \exp\left(lr\left(a\right)\right) = O\left(\exp\left(-wn^{1/3-2\beta}\right)\right) \to 0 \text{, as } n \to \infty \ ,$$

where w is a positive constant.

Now define $b = \bar{X} + \varphi_n n^{-1/2}$ and in a similar manner to the proof scheme above we have

$$\int_{b}^{X_{(n)}} e^{lr(\theta)}\pi(\theta)d\theta \le \exp(lr(b)) = O\left(\exp\left(-w_1\varphi_n^2\right)\right) \to 0,$$

where w_1 is a positive constant and $n \to \infty$.

Thus we show that $\int_{X_{(1)}}^{X_{(n)}} e^{lr(\theta)}\pi\left(\theta\right)d\theta \cong \int_{\bar{X}-\varphi_n n^{-1/2}}^{\bar{X}+\varphi_n n^{-1/2}} e^{lr(\theta)}\pi\left(\theta\right)d\theta$.

Similarly we have that $\int_{X_{(1)}}^{q_L} e^{lr(\theta)}\pi\left(\theta\right)d\theta \cong \int_{\bar{X}-\phi_n n^{-1/2}}^{q_L} e^{lr(\theta)}\pi\left(\theta\right)d\theta$.

Now we consider the main term $\int_{a}^{b} e^{lr(\theta)}\pi\left(\theta\right)d\theta$ of the marginal distribution defined in (2). This integral consists of the log empirical likelihood ratio function $lr(\theta)$ and we expand $lr(\theta)$ at $\theta = \bar{X}$ using Taylor theorem,

$$lr(\theta) = lr(\bar{X}) + (\theta - \bar{X})\lambda(\bar{X}) + \frac{1}{2}(\theta - \bar{X})^2\left(\left.{}^{d\lambda(u)}\!\big/_{du}\right|_{u=\bar{X}}\right)$$
$$+\frac{1}{6}(\theta - \bar{X})^3\left(\left.{}^{d^2\lambda(u)}\!\big/_{du^2}\right|_{u=\bar{X}}\right) + \frac{1}{24}(\theta - \bar{X})^4\left(\left.{}^{d^3\lambda(u)}\!\big/_{du^3}\right|_{u=\theta+\varpi(\bar{X}-\theta)}\right), \varpi \in [0,1]. \quad (1.7)$$

By virtue of Proposition 10.2.1 in Vexler et al. (2014a), we have

$$\left.\frac{d\lambda(u)}{du}\right|_{u=\bar{X}} = -\frac{n}{n^{-1}\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2} = -\frac{n}{\sigma_n^2}, \left.\frac{d^2\lambda(u)}{du^2}\right|_{u=\bar{X}} = -\frac{2n\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^3/n}{\left[n^{-1}\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2\right]^3} = \frac{2nM_n^3}{\left(\sigma_n^2\right)^3},$$

as well as $d^3\lambda(\theta)/d\theta^3 = O_p(n)$, for $\theta\in[a,b]$. The argument $\bar{X}$ maximizes the function $l(\theta)=\sum_{i=1}^{n}\log p_i$, $l(\bar{X}) = n\log(1/n)$ and then $lr(\bar{X})=0$ as well as $\lambda(\bar{X})=0$.

Using the results above, (1.7) and a Taylor expansion for $nM_n^3(\theta-\bar{X})^3/3(\sigma_n^2)^3$ and $O_p(n)(\theta-\bar{X})^4$ around zero, we have

$$\int_a^b e^{lr(\theta)}\pi(\theta)d\theta = \int_a^b \exp\left[-\frac{n}{2\sigma_n^2}(\theta-\bar{X})^2 + \frac{nM_n^3}{3(\sigma_n^2)^3}(\theta-\bar{X})^3 + O_p(n)(\theta-\bar{X})^4\right]\pi(\theta)d\theta$$
$$= \int \exp\left[-\frac{n}{2\sigma_n^2}(\theta-\bar{X})^2\right]\pi(\theta)d\theta + \frac{nM_n^3}{3(\sigma_n^2)^3}\int(\theta-\bar{X})^3\exp\left[-\frac{n}{2\sigma_n^2}(\theta-\bar{X})^2\right]\pi(\theta)d\theta$$
$$+O_p(n)\int(\theta-\bar{X})^4\exp\left[-\frac{n}{2\sigma_n^2}(\theta-\bar{X})^2\right]\pi(\theta)d\theta. \quad (1.8)$$

Now by virtue of the definition (3), the formula $\frac{\alpha}{2} = \int_{X_{(1)}}^{q_L} h_E(\theta\mid X)d\theta$ can be rewritten as

$$\frac{\alpha}{2} = \frac{\int^{q_L}\exp\left[-\frac{n}{2\sigma_n^2}(\theta-\bar{X})^2\right]\pi(\theta)d\theta}{\int\exp\left[-\frac{n}{2\sigma_n^2}(\theta-\bar{X})^2\right]\pi(\theta)d\theta} + R_n,$$

where

$$R_n = \left\{\int_a^{q_L} e^{lr(\theta)}\pi(\theta)d\theta\int\exp\left[-\frac{n}{2\sigma_n^2}(\theta-\bar{X})^2\right]\pi(\theta)d\theta - \right.$$
$$\left.\int^{q_L}\exp\left[-\frac{n}{2\sigma_n^2}(\theta-\bar{X})^2\right]\pi(\theta)d\theta\int_a^b e^{lr(\theta)}\pi(\theta)d\theta\right\}\left\{\int_a^b e^{lr(\theta)}\pi(\theta)d\theta\int\exp\left[-\frac{n}{2\sigma_n^2}(\theta-\bar{X})^2\right]\pi(\theta)d\theta\right\}^{-1}$$

It is clear that one can use (1.8) and the facts:

(1) $\pi(\theta)=\pi(\bar{X})+(\theta-\bar{X})\pi'(\bar{X})+\frac{1}{2}(\theta-\bar{X})^2\pi''\left(\bar{X}+q\left(\theta-\bar{X}\right)\right), q\in[0,1]$;

(2) $b-a=n^{1/6-\beta}/n^{1/2}$; (3) $\int(\theta-\bar{X})\exp\left[-\frac{n}{2\sigma_n^2}(\theta-\bar{X})^2\right]d\theta=0$ to represent the

remainder term R_n in the form

$$R_n = M_n^3 C_n + O_p\left(n^{-1+\varepsilon}\right),$$

where $C_n = 2n^{-0.5}\left(1+z_{1-\alpha/2}^2/2\right)\varphi\left(z_{1-\alpha/2}\right)/3\sigma_n^3$ and $\varepsilon > 0$.

Combing the above asymptotic approximations, we have

$$\frac{\alpha}{2} = \frac{\int_{X_{(1)}}^{q_L} \exp\left[-\frac{n}{2\sigma_n^2}(\theta-\overline{X})^2\right]\pi\left(\theta\right)d\theta}{\int_{X_{(1)}}^{X_{(n)}} \exp\left[-\frac{n}{2\sigma_n^2}(\theta-\overline{X})^2\right]\pi\left(\theta\right)d\theta} + M_n^3 C_n + O_p\left(n^{-1+\varepsilon}\right). \quad (1.9)$$

Similarly one can show that

$$1-\frac{\alpha}{2} = \frac{\int_{X_{(1)}}^{q_U} \exp\left[-\frac{n}{2\sigma_n^2}(\theta-\overline{X})^2\right]\pi\left(\theta\right)d\theta}{\int_{X_{(1)}}^{X_{(n)}} \exp\left[-\frac{n}{2\sigma_n^2}(\theta-\overline{X})^2\right]\pi\left(\theta\right)d\theta} + M_n^3 C_n + O_p\left(n^{-1+\varepsilon}\right).$$

The proof of Proposition 1 is complete.

Proof of Lemma1.

The proof of Lemma 1 is just a straightforward rearrangement of the Equations in Proposition 1. It shows a similar structure as compared to the classic Bayesian Normal/Normal model. For details see Carlin and Louis (2009).

Proof of Proposition 2.

We assume that $\pi(\theta)$ is a normal density function with mean μ and variance σ^2. We first derive a Lemma which is useful for this proof.

Lemma 2, $\left|u_L - z_{\alpha/2}\right| = O\left(n^{-0.5+\varepsilon}\right)$ where u_L is defined in Proposition 2.

Proof: first note the order of a component in remainder term R_n, $M_n^3 C_n = O_p(n^{-0.5+\varepsilon})$.

Equation (1.9) with the above fact imply

$$\frac{\alpha}{2} = \Phi(u_L) + O_p(n^{-0.5+\varepsilon}) \quad , \quad (1.10)$$

where $\Phi(\cdot)$ is a cumulative distribution of the standard normal random variable.

Now we rearrange equation (1.6) as

$$\Phi(u_L) - \Phi(z_{1-\alpha/2}) = O_p(n^{-0.5+\varepsilon}), \tag{1.11}$$

where $z_{1-\alpha/2}$ is defined as $\Phi(z_{1-\alpha/2}) = \alpha/2$.

We also have the inequality $\Phi(u_L) - \Phi(z_{\alpha/2}) = \frac{1}{\sqrt{2\pi}} \int_{z_{1-\alpha/2}}^{u_L} \exp(-\frac{z^2}{2}) dz \leq \frac{|u_L - z_{\alpha/2}|}{\sqrt{2\pi}}$

Combining equation (1.7) and above result, we have as $n \to \infty$,

$|u - z_{\alpha/2}| = O_p(n^{-0.5+\varepsilon})$.

This shows the proof of Lemma 2.

Base on Lemma 1, Equation (1.9) becomes

$$\frac{\alpha}{2} = \Phi(u_L) + M_n^3 C_n + O_p\left(n^{-1+\varepsilon}\right). \tag{1.12}$$

Now we expand function $\Phi(u_L)$ using the Taylor theorem with respect to u_L around $u_L = z_{1-\alpha/2}$, we have

$$\Phi(u_L) = \Phi(u_L)_{u_L = z_{1-\alpha/2}} + \Phi(u_L)'_{u_L = z_{1-\alpha/2}} (u_L - z_{\alpha/2}) + O_p(n^{-1+\varepsilon}). \tag{1.13}$$

Combing Equations (1.13) and (1.12), we obtain

$$\frac{\alpha}{2} = \Phi(u)_{u=z_{\alpha/2}} + \Phi(u)'_{u=z_{\alpha/2}} (u - z_{\alpha/2}) + M_n^3 C_n + O_p(n^{-1+\varepsilon}). \tag{1.14}$$

Then we have the expression for q_L as

$$q_L = \frac{\sigma_n^2 \mu + n\sigma^2 \overline{X}}{\sigma_n^2 + n\sigma^2} - z_{1-\alpha/2} \sqrt{\frac{\sigma_n^2 \sigma^2}{\sigma_n^2 + n\sigma^2}} + \frac{M_n^3}{3n\sigma_n^2}\left(2 + z_{1-\alpha/2}^2\right) + o_p\left(n^{-1}\right).$$

In a similar manner one can show that the expression for q_U as

$$q_U = \frac{\sigma_n^2 \mu + n\sigma^2 \overline{X}}{\sigma_n^2 + n\sigma^2} + z_{1-\alpha/2} \sqrt{\frac{\sigma_n^2 \sigma^2}{\sigma_n^2 + n\sigma^2}} + \frac{M_n^3}{3n\sigma_n^2}\left(2 + z_{1-\alpha/2}^2\right) + o_p\left(n^{-1}\right).$$

The proof of Proposition 2 is complete.

Proof of Propositions 3-9.

The proof of Proposition 3 is omitted since it directly follows from the application of Slutsky’s theorem and an Edgeworth expansion technique.

One can use the proof schemes of Propositions 1 and 2 to show Propositions 4-9 in a similar manner.

R-Code

```
################################################################
########## R code to calculate the proposed confidence interval (CI) estimation
###########                 for the Monte Carlo simulations
################################################################

#The sample size ,significant level alpha and library in R

   library("emplik")
   n<- 50 ; alpha<- 0.05


# Assume that the baseline data distribution is log-normal with mean zero and variance 1
# and prior distribution is normal distribution with mean zero and variance 1.#
# Generate a sample of n centered random variables from the baseline data distribution#

   x<- rlnorm(n,0,1)-exp(0.5)
```

#Create function integ which equals to numerator $e^{lr(\theta)}\pi(\theta)$ in Equation (2)

```
integ<-function(u){
   R<-exp((el.test(x,u)$'-2LLR')*(-0.5))*dnorm(u,0,1)
   return(R)      }

# Calculate denominator in Equation (2) and Split the integral to improve accuracy

   c<-min(x); a<- mean(x)-n^(-0.5+1/6); b<- mean(x)+n^(-0.5+1/6); d<-max(x)

   dem2<-integrate(Vectorize(integ),c,2*a,stop.on.error=FALSE)$value+
             integrate(Vectorize(integ),2*a,a,stop.on.error=FALSE)$value+
             integrate(Vectorize(integ),a,mean(x),stop.on.error=FALSE)$value+
             integrate(Vectorize(integ),mean(x),b,stop.on.error=FALSE)$value+
             integrate(Vectorize(integ),b,2*b,stop.on.error=FALSE)$value+
             integrate(Vectorize(integ),2*b,d,stop.on.error=FALSE)$value
```

Create functions given in Equation (3)

```
f1<-function(q){
F<-((integrate(Vectorize(integ),lower=c,upper=a/2,stop.on.error=FALSE)$value+
integrate(Vectorize(integ),lower=a/2,upper=a,stop.on.error=FALSE)$value+
integrate(Vectorize(integ),lower=a,upper=q,stop.on.error=FALSE)$value)/dem2-alpha/2
)
return(F)        }
f2<-function(q){
F<- ((integrate(Vectorize(integ),lower=q,upper=b,stop.on.error=FALSE)$value+
      integrate(Vectorize(integ),lower=b,upper=2*b,stop.on.error=FALSE)$value+
      integrate(Vectorize(integ),lower=2*b,upper=d,stop.on.error=FALSE)$value)/dem2
      -alpha/2)
    return(F)    }
```

Calculate (1-alpha)100% Data-driven equal-tailed CI

```
c11<-uniroot(f1,c(c,d))$root
c12<-uniroot(f2,c(c,d))$root
print(c(c11,c12))
```

#Create functions given in Equation (10)

```
integ1<- Vectorize(integ)
f2<-function(u){
integrate(integ1,lower=u[1],upper=u[2],stop.on.error=FALSE)$value/dem2 }
Model1 <- function(u){ F1<- ( integ1(u[1])- integ1(u[2]) )
        F2<- (f2(u)-1+alpha )
             FF<- (F1^2+F2^2)
 return(FF)                }
```

Calculate (1-alpha)100% Data-driven HPD CI

```
solu3<- optim(c(c11,c12),f=Model1)$par
print(solu3)
```

Calculate (1-alpha)100% classical CI

```
c21<- mean(x)-zv*sqrt(var(x)/n)
c22<- mean(x)+zv*sqrt(var(x)/n)
print(c(c21,c22))
```

ADDITIONAL MONTE CARLO RESULTS

In this section, using the framework described in Section 3, we demonstrate numerical comparisons between the proposed nonparametric approach and the following methods:

(1) The inverse Edgeworth expansion based method proposed by Hall (1983).

(2) The parametric Bayesian CI estimation.

(3) A frequentist method for improved CI estimation of the log-normal mean. This CI estimation based on log-normally distributed data is described in Zhou and Gao (1997). In this aspect Zhou and Gao (1997) suggested to use the Cox's method to improve the CI estimation. In our study we apply the Cox's approach (see for details Zhou and Gao 1997).

(4) The classical EL CI estimation (Owen 2001).

We present the results of this limited Monte Carlo study in Tables 1 and 2.

Table 1. The Monte Carlo coverage probabilities (CP) and average lengths (LG) for the CI estimation of the mean. The notations ET and HPD represent the proposed equal-tailed and highest posterior CI estimation; ELR represents the classical empirical likelihood ratio CI estimation; H represents the Hall (1983)'s method. C represents the well-known Cox's method CI estimation which is based on the maximum likelihood technique.

$X_1,\ldots,X_n \sim Lognorm(0,2)$, Prior: $\pi \sim N(\exp(2),1)$										
n	ET		HPD		ELR		H		C	
	CP	LG	CP	LG	CP	LG	CP	LG	CP	LG
7	63.97%	3.02	64.18%	3.02	50.96%	19.38	56.77%	22.70	88.38%	700
15	81.77%	3.50	81.77%	3.50	63.84%	17.17	62.53%	17.57	91.92%	181.44
25	87.26%	3.59	87.20%	3.59	69.41%	16.60	64.98%	16.24	92.41%	55.60
50	92.82%	3.59	92.42%	3.59	72.60%	14.32	67.95%	13.57	93.75%	21.22
100	95.77%	3.51	95.77%	3.51	79.06%	11.69	73.05%	10.89	92.87%	12.16
$X_1,\ldots,X_n \sim Lognorm(0,\ 1.5)$, Prior: $\pi \sim N(\exp((1.5^2)/2),\ 1)$										
n	ET		HPD		ELR		H		C	
	CP	LG	CP	LG	CP	LG	CP	LG	CP	LG
7	70.13%	2.43	69.53%	2.42	63.57%	5.59	68.43%	6.50	88.70%	95.99
15	82.37%	2.52	81.13%	2.51	74.04%	5.37	73.32%	5.49	91.81%	13.52
25	87.39%	2.45	86.08%	2.43	80.39%	4.52	77.45%	4.43	92.42%	7.42
50	91.37%	2.27	90.45%	2.24	85.00%	3.77	81.49%	3.60	94.97%	4.35

100	92.46%	2.02	90.95%	1.99	86.42%	3.03	80.89%	2.85	94.47%	2.83
$X_1,...,X_n \sim Lognorm(0,\ 1)$, Prior: $\pi \sim N(\exp(0.5),\ 1)$										
n	ET		HPD		ELR		H		C	
	CP	LG	CP	LG	CP	LG	CP	LG	CP	LG
7	77.97%	1.53	77.13%	1.52	74.50%	2.12	79.70%	2.44	90.04%	5.98
15	86.79%	1.44	86.52%	1.42	83.89%	1.76	84.53%	1.80	93.29%	2.46
25	90.15%	1.30	89.73%	1.27	87.49%	1.50	86.00%	1.48	94.31%	1.76
50	90.34%	1.07	90.70%	1.05	89.13%	1.15	86.70%	1.11	92.78%	1.17
100	93.20%	0.83	93.54%	0.81	93.20%	0.83	91.16%	0.80	94.56%	0.81
$X_1,...,X_n \sim Lognorm(0,\ 1)$, Prior: $\pi \sim N(\exp(0.5),\ 0.5)$										
n	ET		HPD		ELR		H		C	
	CP	LG	CP	LG	CP	LG	CP	LG	CP	LG
7	79.34%	1.19	79.24%	1.19	73.84%	2.12	78.23%	2.44	89.51%	5.29
15	89.66%	1.14	88.80%	1.13	83.74%	1.73	83.77%	1.78	92.52%	2.45
25	92.58%	1.06	92.00%	1.05	87.38%	1.50	86.22%	1.48	94.22%	1.76
50	94.13%	0.91	94.20%	0.90	91.48%	1.13	89.18%	1.09	94.27%	1.18
100	94.52%	0.74	94.00%	0.73	92.73%	0.82	90.72%	0.79	94.18%	0.82
$X_1,...,X_n \sim Lognorm(0,\ 1)$, Prior: $\pi \sim N(\exp(0.5)+1,\ 1)$										
n	ET		HPD		ELR		H		C	
	CP	LG	CP	LG	CP	LG	CP	LG	CP	LG
7	74.50%	1.66	75.20%	1.65	73.37%	2.08	78.40%	2.39	88.83%	4.99
15	83.83%	1.61	85.03%	1.59	83.90%	1.81	83.47%	1.86	91.90%	2.49
25	88.13%	1.45	88.63%	1.43	87.37%	1.50	85.77%	1.48	93.67%	1.74
50	89.97%	1.17	91.47%	1.14	90.37%	1.12	88.63%	1.08	94.37%	1.18
100	92.12%	0.90	93.90%	0.88	93.07%	0.84	90.68%	0.80	95.03%	0.81

Table 2. The Monte Carlo coverage probabilities (CP) and average lengths (LG) for the CI estimation of the mean. The notations ET and HPD represent the proposed equal-tailed and highest posterior CI estimation; PBM represents the parametric Bayesian CI estimation given the known data distribution.

$X_1,...,X_n \sim Norm(0,\ 1)$, Prior: $\pi \sim N(0,\ 1)$						
n	ET		HPD		PBM	
	CP	LG	CP	LG	CP	LG
7	86.81%	1.18	87.44%	1.18	95.36	1.34
15	92.73%	0.96	92.97%	0.95	95.37%	0.98
25	93.71%	0.77	93.87%	0.77	95.23%	0.77

50	95.28%	0.56	95.21%	0.56	94.85%	0.55
100	94.85%	0.39	94.74%	0.39	94.51%	0.39
$X_1,\ldots,X_n \sim Norm(0,\ 1)$, Prior: $\pi \sim N(0,\ 0.5)$						
n	ET		HPD		PBM	
	CP	LG	CP	LG	CP	LG
7	88.43%	1.03	89.06%	1.02	97.02%	1.15
15	94.03%	0.87	94.07%	0.87	97.14%	0.90
25	95.70%	0.72	95.63%	0.72	96.83%	0.73
50	94.71%	0.54	94.87%	0.54	94.71%	0.53
100	95.97%	0.39	95.97%	0.39	96.78%	0.38

In the considered MC scenarios, the data-driven CI estimation outperforms the approach based on Hall's approximations. Perhaps, this event is a result of the fact that in order to obtain the Hall's CI it is required to estimate several unknown parameters within the asymptotic approximations. The estimators of the parameters can be very biased when skewed data are used. The proposed CI approach demonstrates better CPs and LPs than those provided by the classical EL method. In the context of the comparisons with (2) and (3), the new distribution-free CI estimation shows results that are very close to outputs of the parametrical methods.

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