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An empirical likelihood ratio based goodness-of-fit test for Inverse Gaussian distributions

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ABSTRACT

The Inverse Gaussian (IG) distribution is commonly introduced to model and examine right skewed data having positive support. When applying the IG model, it is critical to develop efficient goodness-of-fit tests. In this article, we propose a new test statistic for examining the IG goodness-of-fit based on approximating parametric likelihood ratios. The parametric likelihood ratio methodology is well-known to provide powerful likelihood ratio tests. In the nonparametric context, the classical empirical likelihood (EL) ratio method is often applied in order to efficiently approximate properties of parametric likelihoods, using an approach based on substituting empirical distribution functions for their population counterparts. The optimal parametric likelihood ratio approach is however based on density functions. We develop and analyze the EL ratio approach based on densities in order to test the IG model fit. We show that the proposed test is an improvement over the entropy-based goodness-of-fit test for IG presented by Mudholkar and Tian (2002). Theoretical support is obtained by proving consistency of the new test and an asymptotic proposition regarding the null distribution of the proposed test statistic. Monte Carlo simulations confirm the powerful properties of the proposed method. Real data examples demonstrate the applicability of the density-based EL ratio goodness-of-fit test for an IG assumption in practice.

1. Introduction

The Inverse Gaussian (IG) distribution has a probability density function of the form

$$f(x|\mu,\lambda)=\left(\frac{\lambda}{2\pi x^3}\right)^{1/2}\exp\left\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right\},\quad x>0,$$

where μ and λ are parameters. The IG(μ, λ) distribution is extensively known for modeling and analyzing right skewed data with positive support across several different fields of science, e.g., demography, electrical networks, meteorology, hydrology, ecology, entomology, physiology, and cardiology (see, for example, Folks and Chhikara, 1978, 1989; Bardsley, 1980; Seshadri, 1993, 1999; Johnson et al., 1994; Barndorff-Nielsen, 1994). Given the utility of this distribution, it is meaningful to develop a corresponding goodness-fit-test, which has satisfactory statistical properties. Towards this end, we propose constructing a "distribution-free" goodness-of-fit test for the IG distribution, which is based on approximating the appropriate parametric likelihood ratio test statistic.

The parametric likelihood approach is a powerful statistical tool, which provides optimal statistical tests under well-known conditions; e.g., see Lehmann and Romano (2005); Vexler and Wu (2009); Vexler et al. (2010) and Vexler and

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Tarima (2010). By virtue of the Neyman-Pearson lemma, the likelihood ratio f_{H_1}/f_{H_0}, where f_{H_1} and f_{H_0} correspond to the likelihoods under hypotheses H_1 and H_0, respectively, is the most powerful test statistic when f_{H_1} and f_{H_0} are completely known. In the nonparametric context considered in this article, forms of f_{H_1} and f_{H_0} are unknown, but are estimable. In this case, the Neyman-Pearson lemma motivates us to approximate the optimal likelihood ratios using an empirical approach.

Claeskens and Hjort (2004) introduced a complex method to construct goodness-of-fit tests based on a nonparametric density estimation strategy, which is based on producing test statistics in forms of estimated likelihood ratios. This approach assumes that we can present the density under H_1 as

$$f_{H_1}(x) = f_{H_0}(x)\exp\left\{\sum_{j\in S} a_j\psi_j(x)\right\} \Big/ \int f_{H_0}(x)\exp\left\{\sum_{j\in S} a_j\psi_j(x)\right\} dx,$$

where the orthogonal functions ψ_j satisfy certain conditions. The parameters a_j in the previous expression are estimated via the maximum likelihood method, when $j \in S_n$, with the set S_n that approximates S, a subset of the natural integer numbers. Since the technique of Claeskens and Hjort (2004) requires complex evaluations of various parameters and the selection of orthogonal functions, the corresponding test should be based on large samples in order to have the appropriate type I error control while in turn having good power properties.

As an alternative to the strategy outlined above, we develop a relatively simple approach that approximates the most powerful likelihood ratio goodness-of-fit test for the IG distribution. The method is efficient and utilizes adapted principles of the empirical likelihood (EL) methodology. The EL approach allows investigators to employ methods with properties that are close, in the asymptotic sense, to those of parametric likelihood techniques without having to assume a parametric form for the likelihood functions under either hypothesis H_1 or H_0 (e.g., Lazar, 2003; Qin and Lawless, 1994; Owen, 2001; Vexler et al., 2010; Vexler et al., 2009; Yu and Vexler et al., 2009). The main advantage of the EL approach is that it is based on the maximum likelihood methodology when given a set of well-defined empirical constraints. The EL function of n i.i.d. observations $X_1,\ldots,X_n$ has the form of $L_p = \prod_{i=1}^n p_i$, where the p_i's, $i=1,\ldots,n$, maximize L_p and satisfy empirical constraints corresponding to hypotheses of interest. This approach is based on the likelihood methodology involving cumulative distribution functions $F(x)$. In this case, the likelihood has the form of $\prod_{i=1}^n \{F(X_i) - F(X_i^-)\}$ (see, for details, Owen, 2001). For example, if the null hypothesis is $H_0 : E(X) = 0$, then the values of p_i's in L_p should be chosen to maximize L_p given $\sum_{i=1}^n p_i = 1$ and $\sum_{i=1}^n p_i X_i = 0$, where the constraint $\sum_{i=1}^n p_i X_i = 0$ is an empirical version of $E(X)=0$. The components p_i's of the EL can be obtained using Lagrange multipliers.

Vexler and Gurevich (2010), as well as Gurevich and Vexler (2010), utilized the main idea of the classical EL methodology to approximate parametric likelihood ratios. The authors proposed a nonparametric approach based on approximate density functions. In this article, we derive the EL ratio test for the IG distribution, using the density-based and distribution-free likelihood approach by Vexler and Gurevich (2010). Following Mudholkar and Tian (2002), we transform the observations to be $Y_i = 1/\sqrt{X_i}$, $i = 1,\ldots,n$, and present the likelihood function under the alternative hypothesis in the form of $L_f = \prod_{i=1}^n f_i$, where $f_i = f(Y_{(i)})$, f is a density function, $Y_{(i)}$ is the i-order statistic based on $Y_1,\ldots,Y_n$. Values of f_i can be estimated by maximizing L_f given an empirical version of the constraint $\int f(u)du = 1$. The nonparametric likelihood method mentioned above is then utilized for the purpose of developing our goodness-of-fit test statistic. Since the proposed test statistic approximates the most powerful parametric likelihood ratio test, the density-based EL ratio test is shown to have very efficient characteristics. We also demonstrate the proposed test statistic improves upon the decision rule of Mudholkar and Tian (2002) and maintains good type I error control for finite samples.

In Section 2, we create the EL test based on densities for the IG. Here, theoretical propositions depict properties of the test. A Monte Carlo study of the power of the test is presented in Section 3. Section 4 is given to real data examples employing the proposed test. Section 5 consists of concluding remarks.

2. Method

In this section, we derive the EL ratio goodness-of-fit test based on approximating the densities for the IG distribution under the null and alternative hypotheses. The proposed test is shown to be consistent as $n \to \infty$. Consider the problem of testing the composite hypothesis that $X_1,\ldots,X_n$ are distributed as IG with unknown parameters μ and λ. Mudholkar and Tian (2002) presented an entropy characterization property of the IG family. The authors constructed a goodness-of-fit test for the IG distribution based on a test statistic involving the sample entropy. Their test statistic contains an integer parameter with unknown optimal values. The power of the entropy-based test statistic is strongly dependent on values of the unknown parameter. Mudholkar et al. (2001) also developed a goodness-of-fit test for the IG model by using a characterization theorem for the IG distribution for which it is known that $\overline{X} = \sum_{i=1}^n X_i/n$ and $V = \{\sum_{i=1}^n (1/X_i - 1/\overline{X})\}/n$ are independent when the random sample $(X_1, X_2, \ldots, X_n)$ is from an IG population.

To approximate the optimal parametric likelihood ratio, we note the following issues. In the context of testing for an IG, we must assume that the null density function, say f_{H_0}, of $Y_i = (\sqrt{X_i})^{-1}$ is known up to parameters μ and λ, whereas under the alternative hypothesis, Y_i has a completely unknown distribution, f_{H_1}. In this case, the maximum likelihood estimation can be applied to evaluate the unknown parameters μ and λ approximating the H_0-likelihood function. However, the H_1-likelihood function, say $f_{H_1}(y_1,\ldots,y_n)$, should be approximated nonparametrically. Towards this end, we propose to apply a density-based EL technique (e.g., Vexler and Gurevich (2010)). According to this methodology, which is based on densities rather than the classical maximum EL methodology based on the empirical distribution function, we derive approximation of the parametric likelihood $\prod_{i=1}^n f(Y_i)=\prod_{i=1}^n f_i,\quad i=1,\dots,n,$ where $f_i=f(Y_{(i)})$ with $Y_{(i)}=1/\sqrt{X_{(n-i+1)}},\quad X_{(1)}\le X_{(2)},\dots,\le\quad X_{(n)}$. (The transformation $Y_{(i)}=1/\sqrt{X_{(n-i+1)}}$ was proposed by Mudholkar and Tian (2002) in the context of the entropy-based test for the IG.) In the following section, we present the method for approximating the H_1-parametric likelihood function $\prod_{i=1}^n f_i,\quad i=1,\dots,n$ with its empirical counterpart.

2.1. Density-based EL ratio goodness-of-fit test

The distribution-free EL ratio tests are comparable asymptotically with the powerful parametric likelihood ratio tests over variety of statistical inference problems. In Section 1, we outlined the classical EL approach, which has been dealt with extensively in the literature (e.g., Lazar, 2003;; Qin and Lawless, 1994; Owen, 2001; Yu and Vexler et al., 2010; Vexler and Gurevich, 2010). In accordance with the EL literature, the EL approach is a cumulative distribution based method. However, the optimal parametric likelihood ratio tests are density-based. Hence, some modifications are necessary for applying this method to the problem at hand.

Utilizing the main idea of the classical EL technique, the density-based EL ratio tests were derived to provide efficient tests for normality and uniformity (Vexler and Gurevich, 2010). Following the maximum EL methodology, we can derive values of $f_i,\quad i=1,\dots,n$, that maximize L_f and satisfy the empirical constraints under the alternative hypothesis H_1. Obviously, values of f_i should be restricted by the equation $\int f(u)du=1$. Thus, we need an empirical form of the constraint $\int f(u)du=1$. This empirical constraint can be obtained by the following lemma.

Lemma 2.1. Let $f(y)$ be a density function. Then

$$\sum_{j=1}^n\int_{Y_{(j-m)}}^{Y_{(j+m)}} f(y)dy=2m\int_{Y_{(1)}}^{Y_{(n)}} f(y)dy-\sum_{k=1}^{m-1}(m-k)\int_{Y_{(n-k)}}^{Y_{(n-k+1)}} f(y)dy-\sum_{k=1}^{m-1}(m-k)\int_{Y_{(k)}}^{Y_{(k+1)}} f(y)dy$$
$$\cong 2m\int_{Y_{(1)}}^{Y_{(n)}} f(y)dy-\frac{m(m-1)}{n},$$

where $Y_{(j-m)}=Y_{(1)}$, if $j-m\le 1$, and $Y_{(j+m)}=Y_{(n)}$, if $j+m\ge n$.

Proof. We outline the proof in appendix.

(In Lemma 2.1, the empirical approximation '$\cong$' means that in the remainder term

$$\sum_{k=1}^{m-1}(m-k)\left(\int_{Y_{(n-k)}}^{Y_{(n-k+1)}} f(y)dy+\int_{Y_{(k)}}^{Y_{(k+1)}} f(y)dy\right)=\sum_{k=1}^{m-1}(m-k)\left(\int_{F(Y_{(n-k)})}^{F(Y_{(n-k+1)})} dy+\int_{F(Y_{(k)})}^{F(Y_{(k+1)})} dy\right)$$

the corresponding distribution function F of Y was replaced by the empirical cumulative distribution function based on $Y_1,\dots,Y_n$, for details see Appendix A.)

By virtue of Lemma 2.1, it is obvious that since

$$\int_{Y_{(1)}}^{Y_{(n)}} f(y)dy\le\int_{-\infty}^{\infty} f(y)dy=1,$$
$$\Gamma_m\equiv\frac{1}{2m}\sum_{j=1}^n\int_{Y_{(j-m)}}^{Y_{(j+m)}} f(y)dy\le 1. \qquad (1)$$

Note that, using the empirical approximation to the remainder term in Lemma 2.1, we have

$$\Gamma_m\cong\int_{Y_{(1)}}^{Y_{(n)}} f(y)dy-\frac{(m-1)}{2n}\le 1-\frac{(m-1)}{2n}.$$

We can also remark that considering the representation in Lemma 2.1, one can empirically estimate Γ_m via

$$\hat{\Gamma}_m=\int_{F_n(Y_{(1)})}^{F_n(Y_{(n)})} dy-\frac{(m-1)}{2n}=1-\frac{1}{n}-\frac{(m-1)}{2n}.$$

Thus, we can obtain that $\Gamma_m\approx 1$ when $m/n\to 0$ as $m,n\to\infty$. For simplicity, by applying the approximate analog to the mean value integration theorem, we can write

$$\int_{Y_{(j-m)}}^{Y_{(j+m)}} f(y)dy\cong(Y_{(j+m)}-Y_{(j-m)})f_j.$$

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Therefore, by virtue of Eq. (1), the empirical constraint under the alternative hypothesis is given by

$$\tilde{\Gamma}_m \equiv \frac{1}{2m}\sum_{j=1}^{n} f_j(Y_{(j+m)}-Y_{(j-m)}) \le 1. \tag{2}$$

Consequently, under the empirical constraint (2), the Lagrangian function of the log EL is

$$\sum_{j=1}^{n} \log f_j + \lambda_1 \left(\frac{1}{2m}\sum_{j=1}^{n} f_j(Y_{(j+m)}-Y_{(j-m)})-1\right), \tag{3}$$

where λ_1 is a Lagrange multiplier. Values of f_j maximizing Eq. (3) given the constraint Eq. (2) satisfy the equation

$$\frac{1}{f_j} + \frac{\lambda_1}{2m}(Y_{(j+m)}-Y_{(j-m)}) = 0 \tag{4}$$

Then, $1+\lambda_1(2m)^{-1} f_j(Y_{(j+m)}-Y_{(j-m)})=0$ and taking summation, we conclude with

$$n+\lambda_1(2m)^{-1}\sum_{j=1}^{n} f_j(Y_{(j+m)}-Y_{(j-m)}) = 0.$$

Since Eq. (2), we have $\lambda_1 = -n$. Finally, by Eq. (4), the target values of $f_1,\ldots,f_n$ have the form of

$$f_j = \frac{2m}{n(Y_{(j+m)}-Y_{(j-m)})}, \quad j=1,\ldots,n, \tag{5}$$

where $Y_{(j-m)} = Y_{(1)}$, if $j-m \le 1$, and $Y_{(j+m)} = Y_{(n)}$, if $j+m \ge n$.

Thus, using the maximum EL method, the likelihood ratio test statistic can be constructed as

$$TK_{nm} = \frac{\prod_{j=1}^{n} \frac{2m}{n(Y_{(j+m)}-Y_{(j-m)})}}{\max_{\theta} \prod_{j=1}^{n} f_{H_0}(Y_{(j)}|\theta)}. \tag{6}$$

The density $f_{H_0}(u|\theta)$ has the form of

$$f_{H_0}(u|\theta) = \left(\frac{2\lambda}{\pi}\right)^{1/2} \exp(-\lambda(u^{-1}-\mu u)^2/2\mu^2),$$

where $\theta = (\mu, \lambda)'$.

When the parameters μ and λ are unknown, the maximum likelihood estimators $\hat{\mu}$ and $\hat{\lambda}$ can be applied to Eq. (6). Now the test statistic can be written as

$$TK_{nm} = \frac{\prod_{j=1}^{n} \frac{2m}{n(Y_{(j+m)}-Y_{(j-m)})}}{\left(\frac{2\hat{\lambda}}{\pi e}\right)^{n/2}}, \tag{7}$$

where $\hat{\lambda} = n\hat{\mu}^2\left[\sum_{i=1}^{n} (Y_i^{-1}-\hat{\mu}Y_i)^2\right]^{-1}$ and $\hat{\mu} = n^{-1}\sum_{i=1}^{n} Y_i^{-2}$.

This test statistic TK_{nm} is equivalent to the entropy-based test statistic that was proposed by Mudholkar and Tian (2002) who arrived at it in a different manner via entropy-based consideration (for details, see Appendix B). The step-by-step derivation of the EL-based method given above demonstrates how the test statistic TK_{nm} is an approximation to the optimal likelihood ratio. Thus, we expect directly that a test based on TK_{nm} will provide highly efficient characteristics. The distribution of the test statistic depends strongly on values of the integer parameter m. To efficiently execute the test based on sample entropy, the optimal values of m should be evaluated. In accordance with Mudholkar and Tian (2002), these optimal values of m can be presented using information regarding the alternative distribution. We take this one step forth and can improve upon the test statistics TK_{nm} in the context of eliminating dependence on the integer parameter m by reconsidering the test construction with respect to the EL concept. The constraint Eq. (1) is taken into account in order to derive the EL ratio test based on densities. However, if for some m, $\Gamma_m = \frac{1}{2m}\sum_{j=1}^{n}\int_{Y_{(j-m)}}^{Y_{(j+m)}} f(y)dy \ge 1$, then $\int_{Y_{(1)}}^{Y_{(n)}} f(y)dy \ge 1$ by virtue of Lemma 2.1, which is inadmissible. Thus, we restrict values of f_j to satisfy $\tilde{\Gamma}_m = \frac{1}{2m}\sum_{j=1}^{n} f_j(Y_{(j-m)}-Y_{(j+m)}) \le 1$, for all m. It is clear that there exists an integer m_0 satisfying

$$\max_{f_1,\ldots,f_n:\tilde{\Gamma}_m \le 1, \text{for all } m} \prod_{i=1}^{n} f_i \le \max_{f_1,\ldots,f_n:\tilde{\Gamma}_{m_0} \le 1} \prod_{i=1}^{n} f_i.$$

Therefore,

$$\max_{f_1,\ldots,f_n:\tilde{\Gamma}_m \le 1, \text{for all } m} \prod_{i=1}^{n} f_i \le \min_{1 \le m_0} \max_{f_1,\ldots,f_n:\tilde{\Gamma}_{m_0} \le 1} \prod_{i=1}^{n} f_i. \tag{8}$$

(Here, if $a > b$, then, $\min(a) > \min(b)$.)

Because the constraint (2) approximates Eq. (1), if $f_j,\quad j=1,\ldots,n$ satisfies $\tilde{\Gamma}_r \le 1$, for some r, we have

$$1 \ge \frac{1}{2r}\sum_{j=1}^{r} f_j(Y_{(j+r)} - Y_{(j-r)}) \cong \frac{1}{2r}\sum_{j=1}^{r}\int_{Y_{(j-r)}}^{Y_{(j+r)}} f(u)du \cong \int_{Y_{(1)}}^{Y_{(n)}} f(u)du$$

$$\cong \frac{1}{2k}\sum_{j=1}^{k}\int_{Y_{(j-k)}}^{Y_{(j+k)}} f(u)du \cong \tilde{\Gamma}_k$$

Hence, $\tilde{\Gamma}_k \lesssim 1$. That is, if $f_1,\ldots,f_n$ satisfy $\tilde{\Gamma}_r \le 1$, we can expect that $f_1,\ldots,f_n$ are subject to $\tilde{\Gamma}_k \le 1$, for $k \ne r$ too. Thus, we can expect that

$$\max_{\substack{f_1,\ldots,f_n\,:\,\tilde{\Gamma}_m \le 1,\\ \text{for all } m}} \prod_{i=1}^{n} f_i \cong \max_{\substack{f_1,\ldots,f_n\,:\,\tilde{\Gamma}_{m_0} \le 1\\ \text{for some } m_0}} \prod_{i=1}^{n} f_i \ge \min_{m_0} \max_{f_1,\ldots,f_n:\tilde{\Gamma}_{m_0} \le 1} \prod_{i=1}^{n} f_i. \tag{9}$$

Eqs. (8) and (9) conclude the approximation to the likelihood and can be defined as

$$\min_{m} \max_{f_1,\ldots,f_n:\tilde{\Gamma}_m \le 1} \prod_{i=1}^{n} f_i$$

The occurrence of the minimum of m into the approximation is strongly justified in the proofs of the proposition, which show the asymptotic consistency of the proposed test. This proposition will be introduced below.

Since in the equation below (1), we have the remainder term $(m-1)/2n$ that should be vanished as $n \to \infty$, we also require that $m = o(n)$ as $n \to \infty$. This conditional bound on m is also mentioned in the literature with respect to proving the consistency of entropy-based tests (e.g., Vasicek, 1976; Tusnady, 1977; Vexler and Gurevich, 2010). Thus, we propose the test statistic in the form of

$$TK_n = \frac{\min_{1 \le m < n^{\delta}} \prod_{j=1}^{n} \frac{2m}{n(Y_{(j+m)} - Y_{(j-m)})}}{\left(\frac{2\hat{\lambda}}{\pi e}\right)^{n/2}}, \tag{10}$$

where $\hat{\lambda} = n\hat{\mu}^2\left[\sum_{i=1}^{n}(Y_i^{-1} - \hat{\mu}Y_i)^2\right]^{-1}$ and $\hat{\mu} = n^{-1}\sum_{i=1}^{n} Y_i^{-2}$. In this article, we will utilize $\delta = 0.5$. (Section 3 shows the power demonstrated by the proposed test is relatively the same for different values of $\delta \in (0, 1)$). Finally, we introduce the test for IG distribution having the form: Reject the null IG hypothesis if $\log(TK_n) > C$, where C is a test threshold. The next proposition depicts the asymptotic consistency of the test.

Proposition 2.1. Under $H_0, n^{-1}\log(TK_n) \xrightarrow{p} 0$,

while, under H_1, if $E(\log f_{H_1}(Y_1))^2 < \infty$, then $n^{-1}\log(TK_n) \xrightarrow{p} E\log\left(\frac{f_{H1}(Y_1)}{f_{H0}(Y_1;\, \boldsymbol{a})}\right) > 0$, as $n \to \infty$, where $\boldsymbol{a} = (E(Y_1^{-2}),\quad 1/[E(Y_1^2) - (E(Y_1^{-2}))^{-1}])$.

Proof. See appendix.

The proof of Proposition 2.1 is based on Proposition 2.2 of Vexler and Gurevich (2010), where the minimum of m into the test statistic is formally justified.

Proposition 2.1. demonstrates that $P_{H_1}(\log(TK_n) > C_\alpha) \underset{n\to\infty}{\to} 1$, where C_α is a critical value satisfying the type I error α when $n \to \infty$. Therefore, the proposed test is consistent (i.e., asymptotic power one).

2.2. Null distribution

Certain lines of research have developed around the asymptotic distribution problems involving Vasicek's entropy estimator and different entropy-based statistics (e.g., Cressie, 1976; Dudewicz and van der Meulen, 1981; Hall, 1984, 1986; Khashimov, 1989; Van Es, 1992). It is generally recognized that the asymptotic distribution of the test statistic, $\log(TK_n)$, which includes an estimate of the nuisance parameter, $\hat{\mu}$ of μ as well as $\hat{\lambda}$ of λ, in the IG case, is analytically difficult. Practical applications motivated us to consider critical values for fixed sample sizes. Thus, in this article, we tabulate the

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critical values for fixed sample sizes using a broad set of Monte Carlo simulations. The asymptotic results which were presented in this section assist to control p-values of the proposed test for large samples.

To tabulate the Monte Carlo percentiles of the null distribution, we conducted the following Monte Carlo experiment. The experiment draws 50,000 replicate samples of the test statistic $\log(TK_n)$ at each sample size n. In this experiment, data were generated from the IG(1,1) distribution. The generated values of the test statistic $\log(TK_n)$ were used to determine the critical values $\log(TK^*_{n,\alpha})$ of the null distribution of $\log(TK_n)$ at the significance level α. The results of the experiment are presented in Table 1.

In order to evaluate the accuracy of the obtained critical values, we depict in Table 2 the estimated type I error control using the 5th percentiles of $\log(TK_n)$ test statistic ($\alpha=0.05$). Towards this end, we generated random samples from the IG

Table 1
The critical values[a], $\log(TK^*_{n,\alpha})$, of the proposed test at the significance level α, i.e., $P_{H_0}\{\log(TK_n)>\log(TK^*_{n,\alpha})\}=\alpha$.

Sample size	α							
	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08
10	7.1106	6.5853	6.2781	6.0392	5.8605	5.7016	5.5559	5.4365
15	8.4823	7.8319	7.4314	7.1291	6.9042	6.7207	6.5589	6.4104
20	9.1504	8.4316	7.9990	7.6968	7.4769	7.2840	7.1336	6.9831
25	9.6813	8.9327	8.5005	8.1984	7.9547	7.7582	7.5782	7.4365
30	10.3156	9.5426	9.1027	8.7600	8.4807	8.2622	8.0810	7.9190
35	10.7687	9.9797	9.5197	9.1735	8.9102	8.6944	8.4981	8.3395
40	11.1102	10.3427	9.8696	9.5152	9.2475	9.0397	8.8636	8.7066
45	11.4799	10.7279	10.2458	9.8953	9.6262	9.3990	9.2114	9.0504
50	11.7863	11.0235	10.5751	10.2299	9.9838	9.7577	9.5647	9.3945
55	12.2253	11.4291	10.9412	10.5814	10.3017	10.0863	9.8780	9.6964
60	12.4767	11.6485	11.1331	10.7850	10.5217	10.3099	10.1081	9.9422
65	12.7206	11.8821	11.4015	11.0314	10.7672	10.5457	10.3398	10.1695
70	13.0735	12.2269	11.7186	11.3325	11.0447	10.7979	10.5989	10.4137
75	13.3515	12.5373	12.0057	11.6373	11.3509	11.1066	10.9014	10.7156
80	13.6866	12.8380	12.3101	11.9135	11.5901	11.3451	11.1249	10.9333
85	13.8912	12.9608	12.5013	12.1090	11.8101	11.5501	11.3309	11.1470
90	14.1715	13.2555	12.7261	12.3453	12.0278	11.7836	11.5773	11.3801
95	14.3226	13.3931	12.8802	12.5106	12.2058	11.9523	11.7369	11.5396
100	14.5004	13.5990	13.0714	12.6594	12.3785	12.1138	11.8898	11.6932
120	15.3873	14.4181	13.8550	13.4787	13.1542	12.8996	12.6652	12.4551
150	16.3815	15.3858	14.8241	14.3910	14.0375	13.7557	13.5141	13.2955
200	17.6377	16.7383	16.1220	15.6491	15.3042	15.0077	14.7280	14.5083
250	19.0457	18.0246	17.3325	16.8719	16.4938	16.1581	15.8500	15.5890
300	20.0134	18.8813	18.1871	17.7225	17.3100	16.9705	16.6827	16.4113

Sample size	α					
	0.09	0.1	0.2	0.25	0.3	0.5
10	5.3403	5.2464	4.5818	4.3537	4.1537	3.5341
15	6.2808	6.1572	5.3583	5.0968	4.8658	4.1954
20	6.8429	6.7233	5.8916	5.6168	5.3909	4.6884
25	7.3021	7.1786	6.3610	6.0822	5.8399	5.0826
30	7.7906	7.6603	6.8005	6.5054	6.2528	5.4606
35	8.2009	8.0782	7.1993	6.8813	6.6158	5.7813
40	8.5511	8.4212	7.4996	7.1925	6.9176	6.0574
45	8.9051	8.7682	7.8237	7.4973	7.2154	6.3309
50	9.2391	9.0930	8.1252	7.7896	7.5094	6.5824
55	9.5344	9.3925	8.3903	8.0455	7.7480	6.7965
60	9.7673	9.6370	8.6481	8.3002	8.0015	7.0258
65	10.0041	9.8645	8.8596	8.5034	8.1955	7.1992
70	10.2722	10.1242	9.1086	8.7367	8.4193	7.3990
75	10.5448	10.3932	9.3044	8.9402	8.6249	7.5767
80	10.7531	10.6066	9.5426	9.1530	8.8196	7.7530
85	10.9747	10.8349	9.7473	9.3604	9.0177	7.8937
90	11.2128	11.0538	9.9233	9.5282	9.1813	8.0560
95	11.3760	11.2151	10.0996	9.7067	9.3485	8.2065
100	11.5174	11.3670	10.2358	9.8300	9.4805	8.3179
120	12.2712	12.0964	10.8948	10.4702	10.0877	8.8112
150	13.1050	12.9199	11.6778	11.2222	10.8089	9.4228
200	14.3100	14.1224	12.7396	12.2268	11.7565	10.1912
250	15.3619	15.1536	13.6324	13.0803	12.5864	10.8283
300	16.1683	15.9364	14.3229	13.7106	13.1711	11.2812

[a] Simulation estimates based on 50,000 replications of data at each n and α.

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Table 2
Type I error control[a] of the proposed $\log(TK_n)$ test: α=0.05.

Sample size	IG(1, 0.5)	IG(1, 2)	IG(1, 4)	IG(1, 8)
10	0.0445	0.0532	0.0601	0.0654
20	0.0424	0.0503	0.0541	0.0573
30	0.0486	0.0534	0.0524	0.0553
40	0.0504	0.0514	0.0529	0.0585
50	0.0446	0.0527	0.0520	0.0489

[a] Simulation estimates based on 10,000 replications.

Table 3
The Monte Carlo Type I errors of the proposed test with critical values obtained by Proposition 2.2 to guarantee α=0.05.

Sample size	Monte Carlo Type I error
100	0.0507
150	0.0621
200	0.0829
250	0.0309
300	0.0459
500	0.0334

populations. A selection of the results is displayed in Table 2. It can be seen that the empirical percentiles given in Table 1 provide an excellent type I error control and thus can be confidently recommended to be used in practice.

In this article, we also present an asymptotic result that insures the appropriate significance level of our test is maintained as $n\to\infty$. The following proposition provides an asymptotic upper bound on the significance level of the test based on the statistic (10). Define $m_0 = an^{1/3}[(\log(n))^{2/3}\log(\log(n))]^{-1}+b$, where a and b are constants.

Proposition 2.2. The significance level of the proposed test satisfies asymptotically ($n\to\infty$) the inequality

$$\lim_{n\to\infty} P_{H_0}\left[(6[m_0]n)^{1/2}\left(\frac{\log(TK_n)}{n}-\log(2[m_0])-\gamma+R_{2[m_0]-1}\right)>C\right]\le 1-\Phi(C),$$

where Φ is the standard normal distribution function, γ=0.5772 is Euler constant, and $R_m=\sum_{j=1}^{m}1/j$; $[d]$ is an integer part of d.

Proof. See appendix.

Note that we chose m_0 to minimize a distance between the asymptotic distributions of $\log(TK_n)$ *and* $\log(TK_{n[m_0]})$, where TK_{nm} is defined by Eq. (7). Regarding the constants a and b, we suggest applying the values a=29.42109 and $b=-29.87852$. These values were obtained empirically based on a broad Monte Carlo study. For example, when n=100, by virtue of the Proposition 2.2, the recommended asymptotically critical value is 12.44114 at α=0.05, that corresponds to the actual type I error, 0.0507 obtained using IG(1,1) random samples. The next table represents the actual type I errors of the proposed test, when the critical values were chosen with respect to Proposition 2.2 for $\alpha=0.05$. The results presented in Table 3 are based on generated samples from IG(1,1).

Table 3 demonstrates that the upper bound obtained by Proposition 2.2 can be used in practice.

3. Power properties

The proposed goodness-of-fit test statistic approximated nonparametrically the optimal likelihood ratio. Thus, we expected our test statistic (10) to provide a powerful test as compared to its competitors. The following Monte Carlo study was conducted to investigate the power properties of the proposed IG goodness-of-fit test for moderate sample sizes. In this study, 10,000 repetitions of data with size n=10, 20, 30, were computed from each of the following populations: exponential with mean 1, uniform [0, 1], Weibull (1, 2) with scale parameter 1 and shape parameters 2, and lognormal (0.5, 1) with mean e, and standard deviation $e\sqrt{e-1}$. The power study follows a similar line that was found in Mudholkar and Tian (2002). Table 4 shows the estimated power of our EL goodness-of-fit test Eq. (10) as compared to other goodness-of-fit tests presented by Mudholkar and Tian (2002), Mudholkar et al. (2001), Edgeman et al. (1988) and by Edgeman (1990). Because of the problem of choosing an optimal m, we represent the test proposed by Mudholkar and Tian (2002) for different values of m as seen in Table 4.

The Monte Carlo study demonstrates the power of the EL goodness-of-fit test is superior to or roughly equal to the Mudholkar and Tian (2002) test with values of m that were selected by Mudholkar and Tian empirically, and given known

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Table 4
An empirical power comparison[a] for the density-based empirical likelihood ratio test at α=0.05.

Distribution	N	$K_{n,m=2}$	$K_{n,m=3}$	$K_{n,m=4}$	$K_{n,m=5}$	Z	KS1	KS2	TK_n
Exponential	10	0.2075[b]	0.1991	0.1499	0.0643	0.0206	0.262	0.280	0.2060
	20	0.4668	0.4675[b]	0.4347	0.3781	0.0226	0.518	0.525	0.4636
	30	0.6347	0.6597[b]	0.6563	0.6169	0.0328	0.654	0.668	0.6446
Uniform(0, 1)	10	0.4768[b]	0.4759	0.3987	0.1857	0.1814	0.342	0.356	0.5078
	20	0.8682	0.8802[b]	0.8645	0.8407	0.426	0.616	0.630	0.8826
	30	0.9687	0.9789[b]	0.9833	0.9772	0.5764	0.776	0.782	0.9815
Weibull(1, 2)	10	0.1251[b]	0.1256	0.1064	0.0428	0.0721	0.06	0.074	0.1354
	20	0.2536	0.2707[b]	0.2332	0.2105	0.1611	0.128	0.080	0.2565
	30	0.3564	0.3895[b]	0.4028	0.3636	0.277	0.168	0.111	0.3847
LogNor(0.5, 1)	10	0.0467[b]	0.0464	0.0437	0.0318	0.0407	0.048	0.055	0.0460
	20	0.0588	0.0535[b]	0.0441	0.0351	0.042	0.068	0.080	0.0541
	30	0.0707	0.0679[b]	0.0591	0.0517	0.0397	0.082	0.111	0.0547

[a] Simulation estimates based on 10,000 replications.
[b] Values corresponded to the optimal m found empirically by Mudholkar and Tian (2002) given known alternatives. $K_{n,m}$—entropy-based test (Mudholkar and Tian, 2002); Z—indepedence characterized test (Mudholkar et al., 2001); KS1—modified Kolmogorov–Smirnov test (Edgeman et al., 1988); and KS2—Kolmogorov–Smirnov test using transformation (Edgeman, 1990).

Table 5
An empirical power evaluation[a] of the proposed test statistic (10) with different δ at α=0.05.

Distribution	n	δ=0.4	δ=0.5	δ=0.6
exp(1)	10	0.2179	0.2156	0.2051
	20	0.4659	0.4648	0.4595
	30	0.6541	0.6459	0.6453
Unif(0, 1)	10	0.4974	0.5128	0.4922
	20	0.8819	0.8820	0.8793
	30	0.9803	0.9810	0.9792
Weibull(1, 2)	10	0.1328	0.1358	0.1311
	20	0.2642	0.2634	0.2589
	30	0.3831	0.3875	0.3795

[a] Simulation estimates based on 10,000 replications.

alternatives (see Remark 4.2 in Mudholkar and Tian (2002)). Table 4 depicts how the selection of m strongly affects the power of the test statistic proposed by Mudholkar and Tian (2002). The wrong choice of m can lead to a 50% reduction in the power of the entropy-based test by Mudholkar and Tian (2002). We investigate the power in the next section utilizing the real data examples.

Remark. The definition (10) of the proposed test statistic includes $\delta \in (0, 1)$. We set up $\delta = 0.5$. To investigate the test statistic with different values of δ, we conducted an extensive Monte Carlo study. Table 5 displays a part of outputs obtained in this The Monte Carlo study. Monte Carlo power of the proposed test was not found to be significantly dependent on the values of δ. This is due in part to the fact that the proposed operator min, which appeared in Eq. (10), has the functional ability to detect well the preferable values of the integer parameter m that in turn is a component of the entropy-based test statistic (7). We can hypothesize that preferable values of the integer parameter m are mostly located below or around $n^{0.5}$. (Similar situations were depicted by Vexler and Gurevich, 2010, when the authors analyzed a density-based empirical likelihood test for normality.) To study this issue analytically, we assume complex asymptotic evaluations related to the proposed test are needed, which requires substantial mathematical details beyond the scope of this article. Towards this end, we plan to address asymptotic distributions of the proposed test statistic in future work.

4. Data examples

In this section, we use the proposed EL goodness-of-fit test described above and the test proposed by Mudholkar and Tian (2002) to evaluate the appropriateness of the IG distribution to data from four different studies that were analyzed in Folks and Chhikara (1978). The data sets were composed of shelflife (days) of a food product (say, Dataset 1), fracture toughness of MIG (metal inert gas) welds (say, Dataset 2), precipitation (inches) from Jug Bridge, Maryland (say, Dataset 3), and runoff amounts at Jug Bridge, Maryland (say, Dataset 4). Table 6 presents the p-values obtained via the EL goodness-of-fit test and the entropy-based test by Mudholkar and Tian (2002) for our example data.

Table 6
Test for the IG based on the data introduced in Folks and Chhikara (1978).

Dataset	Sample size	Test	p-Value
1	26	TK_{26}	0.0063
		$K_{26,\ m=2}$	0.0061
		$K_{26,\ m=3}$	0.0058
		$K_{26,\ m=4}$	0.0090
		$K_{26,\ m=5}$	0.0041
2	19	TK_{19}	0.1791
		$K_{19,\ m=2}$	0.1565
		$K_{19,\ m=3}$	0.1516
		$K_{19,\ m=4}$	0.2663
		$K_{19,\ m=5}$	0.3379
3	25	TK_{25}	0.0099
		$K_{25,\ m=2}$	0.0197
		$K_{25,\ m=3}$	0.0159
		$K_{25,\ m=4}$	0.0098
		$K_{25,\ m=5}$	0.0063
4	25	TK_{25}	0.9271
		$K_{25,\ m=2}$	0.9393
		$K_{25,\ m=3}$	0.9538
		$K_{25,\ m=4}$	0.8738
		$K_{25,\ m=5}$	0.8001

Table 7
Bootstrap type proportion of rejection[a] of the tests for the IG at 5% level of significance.

	Sample size	$K_{n,m=2}$	$K_{n,m=3}$	$K_{n,m=4}$	$K_{n,m=5}$	TK_n
Dataset 1	15	0.7199	0.6655	0.6799	0.7024	0.6796
	20	0.9023	0.8574	0.8468	0.8662	0.8499
Dataset 2	10	0.2961	0.2583	0.2784	0.2226	0.2625
	15	0.5649	0.3762	0.3302	0.3422	0.4059
Dataset 3	15	0.688	0.6486	0.5989	0.6507	0.6389
	20	0.8422	0.8168	0.7961	0.7893	0.8082
Dataset 4	15	0.1898	0.0997	0.0728	0.0767	0.0938
	20	0.2820	0.1318	0.0889	0.0761	0.0948

[a] Bootstrap type estimates based on 10,000 replications.

Based on the results from Table 6, our test and the test by Mudholkar and Tian (2002) provide identical conclusions about the goodness-of-fit test of the IG distribution at $\alpha = 0.05$. The IG distribution is rejected for Datasets 1 and 3. The EL-based goodness-of-fit test indicates that Datasets 2 and 4 are well described by an IG distribution. Note that Folks and Chhikara (1978) did not reject the IG distribution for Dataset 1 using Kolmogorov–Smirmov test. In this case, both the proposed test and the entropy-based test proposed by Mudholkar and Tian (2002) rejected the assumption that Dataset 1 follows an IG distribution. If the rejection level is set at $\alpha = 0.01$, the Mudholkar and Tian (2002) test provides different decisions depending on values of m for Dataset 3. The density-based EL ratio goodness-of-fit test demonstrates high sensitivity in these data examples given non-IG alternatives.

In addition to our illustration, a bootstrap type study was conducted to examine the proposed EL-based goodness-of-fit test and the test by Mudholkar and Tian (2002). From each of Datasets 1, 3 and 4, two samples with the sizes 15 and 20 were randomly selected, respectively, in order to be tested for an IG fit at a 5% level of significance. Two samples of sizes 10 and 15 were randomly selected from Dataset 2, respectively, to be tested for an IG fit at a 5% level of significance. We repeated this strategy 10,000 times calculating the frequencies of the events $\log(TK_{10}) > C_{0.05} = 5.8605$, $\log(TK_{15}) > C_{0.05} = 6.9042$, $\log(TK_{20}) > C_{0.05} = 7.4769$ (the critical values were chosen from Table 1). In a similar manner to the structure above, we examined the test proposed by Mudholkar and Tian (2002). For Dataset 2, the proposed test rejected the IG assumption in 2625 (the case of n=10) and 4059 (the case of n=15) events, while the test proposed by Mudholkar and Tian (2002) rejected the IG assumption in 2583 and 3762 events, respectively. Thus, this indicates that our proposed method is more sensitive as compared to Mudholkar and Tian's test. The results of the bootstrap type studies are presented in Table 7.

Table 7 confirms the practical applicability of the proposed test.

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5. Conclusion

In this article, we have developed the EL ratio methodology based on approximating densities nonparametrically under the alternative hypothesis in order to test for the IG distributions. We have proven that the proposed density-based EL ratio test has the structure that is similar to that of the entropy-based goodness-of-fit test for an IG presented by Mudholkar and Tian (2002). Since the goodness-of-fit test Eq. (10) is a nonparametric approximation to the traditional likelihood ratio test, we anticipated good power characteristics. Utilizing a broad Monte Carlo study, we showed that the proposed density-based EL ratio test is powerful when compared with the known goodness-of-fit tests. Also note that the test presented by Mudholkar and Tian (2002) depends on values of an integer parameter m. The optimal values of the Mudholkar and Tian test are generally only known when information regarding parametric forms for the alternative distributions is available. Utilizing the EL concept, we eliminate this restrictive dependence on the parameter m. Theoretical support for the proposed EL ratio test is obtained by proving consistency of the new test and an asymptotic proposition regarding the null distribution of the density-based EL ratio test statistic. The data examples demonstrated that the proposed test is reasonable when applied to real data. In general, the methodology presented in this paper can be easily adapted to construct different nonparametric tests, approximating the optimal likelihood ratios via the EL concept.

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Appendix A

Proof of Lemma 2.1. Vexler and Gurevich (2010) proved the first equality of Lemma 2.1. Now we obtain the second equation of Lemma 2.1, using the empirical distribution functions F_n, we have

$$\begin{aligned}\sum_{j=1}^{n}\int_{Y_{(j-m)}}^{Y_{(j+m)}} f(y)dy &= 2m\int_{Y_{(1)}}^{Y_{(n)}} f(y)dy-\sum_{k=1}^{m-1}(m-k)\int_{Y_{(n-k)}}^{Y_{(n-k+1)}} f(y)dy-\sum_{k=1}^{m-1}(m-k)\int_{Y_{(k)}}^{Y_{(k+1)}} f(y)dy\\
&=2m\int_{Y_{(1)}}^{Y_{(n)}} f(y)dy-\sum_{k=1}^{m-1}(m-k)(F(Y_{(n-k+1)})-F(Y_{(n-k)}))-\sum_{k=1}^{m-1}(m-k)(F(Y_{(k+1)})-F(Y_{(k)}))\\
&\cong 2m\int_{Y_{(1)}}^{Y_{(n)}} f(y)dy-\sum_{k=1}^{m-1}(m-k)(F_n(Y_{(n-k+1)})-F_n(Y_{(n-k)}))-\sum_{k=1}^{m-1}(m-k)(F_n(Y_{(k+1)})-F_n(Y_{(k)}))\\
&=2m\int_{Y_{(1)}}^{Y_{(n)}} f(y)dy-\frac{m(m-1)}{n},\end{aligned}$$

where F and F_n are the theoretical and empirical distribution functions, respectively.

Proof of Proposition 2.1. The proof of Proposition 2.1 is based on Proposition 2.2 of Vexler and Gurevich (2010). In order to show the consistency of the proposed test, let us check conditions that are present in Vexler and Gurevich (2010):

Conditions for Proposition 2.1

(C1) $E(log f(Y_1))^2<\infty$.

(C2) Under the null hypothesis H_0, we define $|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}|=max(|\mu-\hat{\mu}|,\ |\lambda-\hat{\lambda}|)$. Both $\hat{\mu}$ and $\hat{\lambda}$ are the maximum likelihood estimators of μ and λ, respectively. By a property of the maximum likelihood estimator, a consistent estimator, $|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}|\xrightarrow{p}0$ as $n\to\infty$.

(C3) Under the alternative hypothesis H_1, for $\boldsymbol{a}=(E(Y_1^{-2}),\ 1/[E(Y_1^2)-1/E(Y_1^{-2})])$, we have $\hat{\boldsymbol{\theta}}\xrightarrow{p}\boldsymbol{a}$ as $n\to\infty$ by virtue of the law of large numbers.

(C4) There are open intervals $\Theta_0\subseteq R^2$ and $\Theta_1\subseteq R^2$ containing $\boldsymbol{\theta}$ and $\boldsymbol{a}$, respectively, as well as there exists a function $t(y)$ such that $|h_i(y,\tau)|\le t(y)$, for all $y\in R$, $\tau\in\Theta_0\cup\Theta_1$, $i=1, 2$ and $E(t(Y_1))<\infty$. It is clear that (C4) is satisfied.

Proof of Proposition 2.1. Let us outline the proof of Proposition 2.1, using the proof scheme of Proposition 2.2 of Vexler and Gurevich (2010). Consider the statistic

$$R_n=\frac{1}{n}log\min_{1\le m<\sqrt{n}}\prod_{i=1}^{n}\frac{2m}{n(Y_{(i+m)}-Y_{(i-m)})}=-\max_{1\le m<\sqrt{n}} r_{mn},$$

which is the first component of the test statistic, $\log(TK_n)/n$ and $r_{mn}=\frac{1}{n}\sum_{i=1}^{n} log\left(\frac{n(Y_{(i+m)}-Y_{(i-m)})}{2m}\right)$. From Vasicek's (1976), we have

$$r_{mn}=(2m)^{-1}\sum_{j=1}^{2m} S_j+U_{mn},$$

$$S_j=-\sum_{i=1}^{n} log\left(\frac{F(Y_{(i+m)})-F(Y_{(i-m)})}{Y_{(i+m)}-Y_{(i-m)}}\right)(F_n(Y_{(i+m)})-F_n(Y_{(i-m)})),\quad i\equiv j(mod 2m),$$

$$U_{mn}=\frac{1}{n}\sum_{i=1}^{n} log\left(\frac{n}{2m}(F_n(Y_{(i+m)})-F_n(Y_{(i-m)}))\right),$$

where Y's are followed by the distribution F, and have F_n as the empirical distribution function. Following Vasicek (1976), $S_j\to H(f)$ as $n\to\infty$, $m/n\to 0$ uniformly for all $1\le m<\sqrt{n}$, where $H(f)=-\int_{-\infty}^{\infty} f(y)logf(y)dy=-E_f(logf(Y_1))$ is the entropy of the distribution F with a density function $f(y)=\left(\frac{2\lambda}{\pi}\right)^{1/2}\exp(-\lambda(u^{-1}-\mu u)^2/2\mu^2)$, and E_f is an expectation when Y's are distributed from F. The statistic U_{mn} is a non-positive variable independent of F, and hence $U_{mn}\xrightarrow{p}0$ as $n\to\infty$, $m\to\infty$. Hence,

$$R_n\le -r_{\sqrt{n},n}\xrightarrow{p}E_f(logf(Y_1)),\quad R_n\ge -\max_{1\le m<\sqrt{n}}(2m)^{-1}\sum_{j=1}^{2m} S_j\xrightarrow{p}E_f(logf(Y_1)),\quad as\ n\to\infty.$$

Therefore

$$R_n\xrightarrow{p}E_f(log(f(Y_1))),\quad as\ n\to\infty \tag{A.1}$$

Now, we represent the statistic, TK_n in the form of

$$\frac{1}{n}\log(TK_n)=R_n-\frac{1}{n}\sum_{i=1}^{n}\log(f_{H_0}(Y_i|\boldsymbol{\theta}))+\frac{1}{n}\left(\sum_{i=1}^{n}\log(f_{H_0}(Y_i|\boldsymbol{\theta}))-\sum_{i=1}^{n}\log(f_{H_0}(Y_i|\hat{\theta}_{\boldsymbol{n}}))\right),\quad \text{where } \hat{\boldsymbol{\theta}}_{\boldsymbol{n}}=(\hat{\mu},\hat{\lambda}) \tag{A.2}$$

Since Eq. (A.1) under H_0,

$$R_n\xrightarrow{p}E_{f_{H_0}}(\log(f_{H_0}(Y_1))),\quad \text{as } n\to\infty. \tag{A.3}$$

Since $E(logf(Y_1))^2<\infty$,

$$\frac{1}{n}\sum_{i=1}^{n} log(f_{H_0}(Y_i|\boldsymbol{\theta}))\xrightarrow{p}E_{f_{H_0}}(log(f_{H_0}(Y_1|\boldsymbol{\theta}))) \tag{A.4}$$

Let $\boldsymbol{\theta}=(\mu,\lambda)'$ and θ_i be the ith element of the vector $\boldsymbol{\theta}$.

$$h_i(y,\tau)=\frac{\partial logf_{H_0}(y|\boldsymbol{\theta})}{\partial\theta_i},\quad i=1,2,$$

as well as $|\boldsymbol{\theta}|=\max_{1\le i\le 2}|\theta_i|$, where $\boldsymbol{\theta}=(\theta_1,\theta_2)'$.

By expanding the third term of Eq. (A.2) in Taylor series until the first derivative, we obtain

$$\frac{1}{n}\left\{\sum_{i=1}^{n} log(f_{H_0}(Y_i|\boldsymbol{\theta}))-\sum_{i=1}^{n} log\left(f_{H_0}(Y_i|\hat{\boldsymbol{\theta}}_{\boldsymbol{n}})\right)\right\}\cong\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{2} h_j(Y_i,\hat{\boldsymbol{\theta}}_{\boldsymbol{n}})(\theta_j-\hat{\theta}_{nj}).$$

Since there exists a function $t(y)$ such that $|h_i(y,\tau)|\le t(y)$, for all $y\in R$, $\tau\in\Theta_0\cup\Theta_1$, $i=1,2$, and $E(t(Y_1))<\infty$, where $\Theta_0\subseteq R^2$ and $\Theta_1\subseteq R^2$ are open intervals containing $\boldsymbol{\theta}$ and $\boldsymbol{a}$, respectively,

$$\frac{1}{n}\left\{\sum_{i=1}^{n} log(f_{H_0}(Y_i|\boldsymbol{\theta}))-\sum_{i=1}^{n} log(f_{H_0}(Y_i|\hat{\boldsymbol{\theta}}_{\boldsymbol{n}}))\right\}\cong\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{2} h_j(Y_i,\hat{\boldsymbol{\theta}}_{\boldsymbol{n}})(\theta_j-\hat{\theta}_{nj})$$

$$\ge\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{2}|h_j(Y_i,\boldsymbol{\tau}_i)|(\theta_j-\hat{\theta}_{nj}),$$

where $|\tau_i-\boldsymbol{\theta}|\le|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}_n|$. Since, under the null hypothesis, H_0, $|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}|\xrightarrow{p}0$ as $n\to\infty$, where $\hat{\boldsymbol{\theta}}$ is the maximum likelihood estimator of $\boldsymbol{\theta}$,

$$\frac{1}{n}\left\{\sum_{i=1}^{n} log(f_{H_0}(Y_i|\boldsymbol{\theta}))-\sum_{i=1}^{n} log(f_{H_0}(Y_i|\hat{\boldsymbol{\theta}}_{\boldsymbol{n}}))\right\}\xrightarrow{p}0. \tag{A.5}$$

Therefore, under H_0, Eqs. (A.2)–(A.5) conclude

$$\log(TK_n)/n \xrightarrow{p} 0, \quad as\ n\to\infty \tag{A.6}$$

Under H_1,

$$\log(TK_n)/n = R_n - \frac{1}{n}\sum_{i=1}^{n}\log(f_{H_1}(Y_i)) + \frac{1}{n}\sum_{i=1}^{n}\log\frac{f_{H_1}(Y_i)}{f_{H_0}(Y_i|\boldsymbol{a})} + \frac{1}{n}\sum_{i=1}^{n}\log\frac{f_{H_0}(Y_i|\boldsymbol{a})}{f_{H_0}(Y_i|\hat{\boldsymbol{\theta}}_{\boldsymbol{n}})}. \tag{A.7}$$

Similarly to the proof of results (A.3)–(A.5), we conclude that

$$\log(TK_n)/n \xrightarrow{p} E_f\left(\log\frac{f_{H_1}(Y_1)}{f_{H_0}(Y_i|\boldsymbol{a})}\right) > 0 \text{ as } n\to\infty$$

This and Eq. (A.6) complete the proof of Proposition 2.1.

Proof of Proposition 2.2. Following Song (2000, 2002), we have

$$P_{H_0}(\log(TK_n) > C) \le P_{H_0}(\log(TK_{n[m_0]}) > C) = 1-\Phi\left((6[m_0]n)^{1/2}\left(\frac{\log(TK_n)}{n} - \log(2[m_0]) - \gamma + R_{2[m_0]-1}\right)\right), \quad as\ n\to\infty,$$

since $[m_0]$ satisfies the conditions $[m_0]/\log n\to\infty$ and $[m_0]\log n^{2/3}/n^{1/3}\to 0$ as $n\to\infty$. (Here, $TK_{n,m}$ by Eq. (7).)

Then,

$$\lim_{n\to\infty} P_{H_0}\left[(6[m_0]n)^{1/2}\left(\frac{\log(TK_n)}{n} - \log(2[m_0]) - \gamma + R_{2[m_0]-1}\right) > C\right] \le 1-\Phi(C).$$

We choose $m=m_0$ to minimize the argument of $(6mn)^{1/2}(\log(TK_n)_m/n - \log(2m) - \gamma + R_{2m-1})$, for all appropriate values of m.

Appendix B

The test statistic TK_{nm} can be shown to be equivalent to the test statistic for an IG distribution based on the sample entropy proposed by Mudholkar and Tian (2002). The density-based EL ratio test statistic is defined as

$$TK_{nm} = \frac{\prod_{j=1}^{n}\frac{2m}{n(Y_{(j+m)}-Y_{(j-m)})}}{\left(\frac{2\hat{\lambda}}{\pi e}\right)^{n/2}}.$$

The maximum likelihood estimator $\hat{\lambda}$ of λ can be written in the form of

$$\begin{aligned}
\hat{\lambda} &= n\hat{\mu}^2\left[\sum_{i=1}^{n}(Y_i^{-1}-\hat{\mu}Y_i)^2\right]^{-1}\\
&= n\left[n^{-2}\left(\sum_{i=1}^{n}Y_i^{-2}\right)^2\right]\left(\sum_{i=1}^{n}\left(Y_i^{-2}-2n^{-1}\sum_{i=1}^{n}Y_i^{-2}+n^{-2}Y_i^{-2}\left(\sum_{i=1}^{n}Y_i^{-2}\right)^2\right)\right)^{-1}\\
&= n^{-1}\left(\sum_{i=1}^{n}Y_i^{-2}\right)^2\left(\sum_{i=1}^{n}Y_i^{-2}-2\sum_{i=1}^{n}Y_i^{-2}+n^{-2}\sum_{i=1}^{n}Y_i^{2}\left(\sum_{i=1}^{n}Y_i^{-2}\right)^2\right)^{-1}\\
&= n^{-1}\left(\sum_{i=1}^{n}Y_i^{-2}\right)^2\left(n^{-2}\sum_{i=1}^{n}Y_i^{2}\left(\sum_{i=1}^{n}Y_i^{-2}\right)^2-\sum_{i=1}^{n}Y_i^{-2}\right)^{-1}\\
&= n^{-1}\left(\sum_{i=1}^{n}Y_i^{-2}\right)^2\Big/\left(n^{-2}\sum_{i=1}^{n}Y_i^{2}\left(\sum_{i=1}^{n}Y_i^{-2}\right)^2-\sum_{i=1}^{n}Y_i^{-2}\right)\\
&= \left(\sum_{i=1}^{n}Y_i^{2}/n-n\left(\sum_{i=1}^{n}Y_i^{-2}\right)^{-1}\right)^{-1}.
\end{aligned}$$

By substituting $\hat{\lambda}$ above into TK_{nm}, TK_{nm} is formulated as

$$TK_{nm} = \frac{\prod_{j=1}^{n}\frac{2m}{n(Y_{(j+m)}-Y_{(j-m)})}}{\left(\frac{2\hat{\lambda}}{\pi e}\right)^{n/2}} = \frac{\prod_{j=1}^{n}\frac{2m}{n(Y_{(j+m)}-Y_{(j-m)})}}{\left(\frac{2\left(\sum_{i=1}^{n}Y_i^2/n-n\left(\sum_{i=1}^{n}Y_i^{-2}\right)^{-1}\right)^{-1}}{\pi e}\right)^{n/2}} = \frac{\prod_{j=1}^{n}\left(\frac{2m}{n(Y_{(j+m)}-Y_{(j-m)})}\right)^{-1/n}}{\left(\frac{2\left(\sum_{i=1}^{n}Y_i^2/n-n\left(\sum_{i=1}^{n}Y_i^{-2}\right)^{-1}\right)}{\pi e}\right)^{1/2}}$$

$$= \frac{\prod_{j=1}^{n}\left(\frac{n\left(Y_{(j+m)}-Y_{(j-m)}\right)}{2m}\right)^{1/n}}{\left(\frac{2\left(\sum_{i=1}^{n}Y_i^2/n-n\left(\sum_{i=1}^{n}Y_i^{-2}\right)^{-1}\right)}{\pi e}\right)^{1/2}} = \frac{\prod_{j=1}^{n}\left(\frac{n\left(Y_{(j+m)}-Y_{(j-m)}\right)}{2m}\right)^{1/n}}{\left(\sum_{i=1}^{n}Y_i^2/n-n\left(\sum_{i=1}^{n}Y_i^{-2}\right)^{-1}\right)^{1/2}/2}.$$

The test statistic TK_{nm} is identical to the test statistic, $exp(H_{mn}(f_Y))/(w/2)$, presented by Mudholkar and Tian (2002), utilizing the maximum likelihood estimator of $\xi=(E(Y^2)-1/E(Y^{-2}))^{1/2}$ instead of the sample UMVU estimator, where $H_{mn}(f_y)=\frac{1}{n}\sum_{i=1}^{n} log\{\frac{n}{2m}(Y_{(j+m)}-Y_{(j-m)})\}$, and w is the sample UMVU estimator of ξ, $w^2=\sum_{i=1}^{n}Y_i^2/(n-1)-n^2\left(\sum_{i=1}^{n}Y_i^{-2}\right)^{-1}/(n-1)$.

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