# GUARANTEED TESTING FOR EPIDEMIC CHANGES OF A LINEAR REGRESSION MODEL [1]

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**SUMMARY**

The objective of this paper is to propose and examine a class of generalized maximum likelihood asymptotic power one tests for detection of various types of changes in a linear regression model. The proposed retrospective tests are based on martingales structures Shiryayev-Roberts statistics. This approach is widely known in a sequential analysis of change point problems as an optimal method of detecting a change in distribution. Guaranteed non-asymptotic upper bounds for the significance levels of the considered tests are presented.

Simulated data sets are used to demonstrate that the proposed tests can give good results in practice.

# 1 Introduction

The observed sample is $\{Y_i, x_{1i}, x_{2i}\}_{i=1}^n$. Without loss of generality and for the sake of clarity of exposition, we assume that $x_{1i}$ and $x_{2i}$ are scalar values. Let

$$\begin{aligned} Y_i &= (\beta_{00} + \beta_{01} x_{1i} + \varepsilon_{0i})\, I\{i < \nu\} \\ &\quad + (\beta_{10} + \beta_{11} x_{1i} + \beta_{12} x_{2i} + \varepsilon_{1i})\, I\{\nu \le i < \gamma\} \\ &\quad + (\beta_{00} + \beta_{01} x_{1i} + \varepsilon_{0i})\, I\{i \ge \gamma\},\ \nu < \gamma,\ i = 1, \ldots, n \end{aligned} \tag{1.1}$$

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denote the segmented linear regression model, where $\beta_{km},\ m=0,\dots,k+1,\ k=0,1$ are regression parameters, $x_{1i}, x_{2i}$ are fixed predictors, $I\{\cdot\}$ is the indicator function, $\nu,\ \gamma$ are the change points, $\varepsilon_{0i}, \varepsilon_{1i}$ are independent random disturbance terms with $g_0(u,\theta_0),\ g_1(u,\theta_1)$ densities ($\theta_0$ and $\theta_1$ are parameters) respectively.

The model corresponds to a situation where up to an unknown change point $\nu > 0$, the observations satisfy the linear regression model with parameters $\beta_{00}, \beta_{01}$. Beyond that (if $\nu \le n$), an epidemic state runs from time $\nu$ through $\gamma - 1$ after which (if $\gamma < n$) the normal state is restored.

The case, where $\beta_{12} \neq 0$, has been extensively dealt with in the literature and corresponds to a control problem, where the baseline in-control distribution is independent of an explanatory variable and the post-change out-of-control distribution is not. One application of this problem is found in epidemiological studies, where it is often assumed that an explanatory variable has no effect on a response prior to a certain unknown change point (e.g. Küchenhoff and Carroll (1997), Leuraud and Benichou (2001)). In the present paper the situation where $\beta_{12} = 0$, is discussed.

For other applications of such models in biostatistics and econometrics see, for example, Broemeling and Tsurumi (1987), Hansen (2000), Braun, Braun, and Müller (2000) as well as Koul, Qian and Surgailis (2003).

Formally, the problem examined in this study is a problem of hypothesis testing where

$$\mathbf{H_0}\ :\ \nu > n, \qquad \text{versus} \quad \mathbf{H_1}\ :\ 1 \le \nu \le n;\ \nu, \gamma :\ 0 < \nu < \gamma \text{ are unknown.} \tag{1.2}$$

Note for example: if $\beta_{12} = 0,\ \beta_{10} = \beta_{00} + \delta,\ \beta_{11} = \beta_{01} + \delta$, $\delta$ is a nuisance parameter and $\varepsilon_{0i}, \varepsilon_{1i}$ are normal distributed with zero expectation and variance $\sigma^2$, then we have the problem considered by Yao (1993); if $\beta_{01} = \beta_{11} = \beta_{12} = 0,\ \beta_{10} = \beta_{00} + \delta,\ \theta_0 = \theta_1$ and $g_0 = g_1$, then we have the problem analyzed by Hušková (1995); if $\gamma > n$ then it is a standard change point problem of regression models (e.g. Julious (2001), Gurevich and Vexler (2004)); for other investigations of such problem see, for example, Račkauskas and Suquet (2003) etc.

The paper proposes and examines a class of generalized maximum likelihood asymptotic power one tests for various types of problem (1.1), (1.2). It is widely known in change point literature (e.g. Lai (1995)) that such tests have high power, therefore, evaluation of their significance level is a major issue. Most results dealing with the significance level of the generalized maximum likelihood ratio tests for problem (1.1), (1.2) (even in the simplest cases of independent identically distributed (*i.i.d.*) observations prior to and after the change) are complex and asymptotic ($n \to \infty$) with special conditions on distribution functions of $\varepsilon_{0i}, \varepsilon_{1i}$ and explanatory variables $x_1, x_2$ (e.g. Yao (1993): Remark 3, p. 184).

The main objective of the paper is an extension of the methods developed to solve change point problems in sequential analysis (Yakir, Krieger and Pollak (1999); Lorden and Pollak (2004)) and proposing generalized maximum likelihood asymptotic power one tests for problem (1.2). The proposed tests have a guaranteed non asymptotic upper bound for significance level, that ensures a p-value. Since this upper bound does not depend on predictors $x_1, x_2$, nothing can be assumed about behaviour of $x_1, x_2$. The power of the tests is discussed in this paper.

The paper is organized as follows. Section 2 introduces the simple case of problem (1.1), (1.2), where parameters $\beta_{00}, \beta_{01}, \theta_0$ of the stable state and parameters $\beta_{10}, \beta_{11}, \beta_{12}, \theta_1$ of the epidemic state of model (1.1) are known. The proposed test statistics is based on the Shiryayev-Roberts approach. The analysis of this section is relatively clear, and has basic ingredients for more general cases. Section 3 considers a more complicated case, where the parameters of the stable state are known and the parameters of the epidemic state are unknown. Section 4 presents a situation, where all parameters of model (1.1) are unknown. The standard form of problem (1.1), (1.2), where $\beta_{12} = 0$, is presented in Section 5. Section 6 represents the results of Monte Carlo simulations.

# 2 Simple Case of the Problem

We consider model (1.1) and test hypothesis (1.2), where $\beta_{00}, \beta_{01}, \theta_0,\ \beta_{10}, \beta_{11}, \beta_{12}, \theta_1$ are known.

Let $P_0$ denote the probability measure under $H_0$ and $P_{km}$ denote the probability measure under $H_1$ with $\nu = k, \gamma = m$. Likewise, let $E_0$ and $E_{km}$ denote expectation under $P_0$ and $P_{km}$, respectively. Denote the likelihood ratio

$$\Lambda_{km} \quad = \quad \prod_{i=k}^{m} \frac{g_1(Y_i - \beta_{10} - \beta_{11}x_{1i} - \beta_{12}x_{2i}, \theta_1)}{g_0(Y_i - \beta_{00} - \beta_{01}x_{1i}, \theta_0)}. \tag{2.3}$$

The standard test statistics for problem (1.2) are some modifications of CUSUM statistics, for instance, $\max_{1\le m\le n} \max_{1\le k\le m} \Lambda_{km}$ (e.g. Yao (1993)). In sequential analysis of solving change point problems there are methods related to substitution of maximum of likelihood ratios by the Shiryaev-Roberts statistics. Hence, that leads us to the Shiryaev-Roberts change point detection policy aimed to obtain guaranteed characteristics of procedures (e.g. Pollak (1987)).

Following this remark, we propose the following test: reject $H_0$ iff

$$\frac{1}{n} \max_{1\le m\le n} R_m \quad > \quad C, \tag{2.4}$$

where $R_m = \sum_{k=1}^{m} \Lambda_{km}$, $\Lambda_{km}$ by (2.3) and the threshold $C > 0$.

**Significance level of the test.** In Proposition 2.1 we obtain a guaranteed non-asymptotic upper bound for the significance level of the proposed test.

**Proposition 2.1** *The significance level $\alpha$ of the test satisfies:*

$$\alpha \equiv P_0 \left\{ \max_{1\le m\le n} R_m > nC \right\} \le 1/C.$$

**Proof.** Define a stopping rule $N_n(A) = \min\{\tau(A), n\}$, where for $A > 0$,

$$\tau(A) \quad = \quad \inf\{l > 1 : \quad R_l \ge A\} \ \ (= \infty \quad \text{if no such } l \text{ exists}). \tag{2.5}$$

It is clear that

$$\alpha \quad = \quad P_0 \left\{ R_{N_n(nC)} > nC \right\} \le \frac{E_0 R_{N_n(nC)}}{nC}. \tag{2.6}$$

As it is well-known (e.g. Pollak (1987)) the sequence $\{R_m - m\}$ is the martingale under $P_0$, so that $E_0 R_{N_n(nC)} = E_0 N_n(nC) \le n$. Combining this result with (2.6) completes the proof of Proposition 2.1.

That is, we have the upper bound (that does not involve $n$ and is independent of different conditions on $x_{1i}, x_{2i},\ i = 1, \ldots, n$) for the significance level of test (2.4): $\alpha \leq 1/C$. Thus, selecting $C = 1/\alpha$ determines a test with a level of significance that does not exceed $\alpha$.

In order to illustrate the closeness of the true significance level to the upper bound we consider the following asymptotic result related to a case, where $\varepsilon_{0i}, \varepsilon_{1i}$ are independent random variables from an exponential family:

$$g_1(u, \theta_1) = \exp(\theta_1 u - \psi(\theta_1)) g_0(u, \theta_0),\ \theta_1 \in \Omega,$$

where $\Omega$ is an interval on which $\psi(.)$ is finite, $\theta_1 \psi'(\theta_1) - \psi(\theta_1) < \infty$. Since the parameters of the model are known, we assume, without loss of generality, that $\beta_{00} = \beta_{01} = \beta_{10} = \beta_{11} = \beta_{12} = 0$.

**Proposition 2.2** *Let* $n^* = n^*(C)$ *be a sequence that satisfies* $\log(C)/n^* \to 0$ *as* $C \to \infty$ *and* $P_{1\infty}$ *is nonlattice, then*

$$\lim_{C \to \infty} C P_0 \left\{ \max_{1 \leq m \leq n^*} R_m > n^* C \right\} = \int_0^\infty e^{-x} dH(x) \leq 1,$$

*where* $H(x)$ *the asymptotic distribution, under the regime* $P_{1\infty}$*, of the overshoot* $\sum_{i=1}^{M(A)} (\theta_1 \varepsilon_{1i} - \psi(\theta_1)) - \log(A)$*,* $M(A) = inf \left\{ l :\ \exp\left( \sum_{i=1}^{l} (\theta_1 \varepsilon_{1i} - \psi(\theta_1)) \right) \geq A \right\}$*,* $A \to \infty$*.*

**Proof.** The proof is based on Lemma 1 by Yakir (1995), p. 274. According to the lemma, if the conditions of Proposition 2.2 are satisfied we have

$$\lim_{A \to \infty} \frac{P_0\{\tau(A) \leq m\}}{P_0\{M(A/m) \leq m\}} = 1,$$

where $\tau(A)$ is defined in (2.5), $A = Cn^*$, $m = n^*$. Applying this and the known results of the renewal theory regarding overshooting (Woodroofe (1982) or Yakir (1995), p. 276) will lead to the asymptotic conclusion of Proposition 2.2.

**Remark 2.1.** For the nonce, we assume the problem (1.1), (1.2) in the form, which is one of the variations considered by Yao (1993). Let $\beta_{00}$, $\beta_{01}$, $\beta_{10}$, $\beta_{11}$ and $\beta_{12}$ be zero as well as suppose that $\varepsilon_{0i}$, $\varepsilon_{1i}$ are independent normal random variables with variance 1 and means $\theta_0$ and $\theta_1$, respectively. Without loss of generality it can be assumed that $\theta_0 = 0$ (since instead of $Y_i$ we can consider a reparametrization of the model $Y_i - \theta_0$) and $\theta_1 > 0$. At this rate, we have

$$g_1(u, \theta_1) = \frac{1}{(2\pi)^{1/2}} \exp\left(-\frac{(u-\theta_1)^2}{2}\right) = \exp\left(\theta_1 u - \frac{\theta_1^2}{2}\right) g_0(u, \theta_0).$$

With this notation, if the sample size $n^*$ and the threshold $C$ are such that $\log(C)/n^* \to 0$ as $C \to \infty$ then the conditions of Proposition 2.2 are satisfied. By applying Theorem 6.2 of Woodroofe (1982) to the result of Proposition 2.2, we obtain for $C \to \infty$

$$P_0\left\{\max_{1\le m\le n^*} R_m > n^* C\right\} \sim h(\theta_1)C, \quad h(x) = 2\frac{1}{x^2}\exp\left(-2\sum_{i=1}^{\infty}\frac{1}{i}\Phi\left(-\frac{1}{2}xi^{1/2}\right)\right),$$

where $\Phi$ denotes the standard normal distribution function. Note that standard statistics for testing (1.2), in this simplest case, have some large deviation approximations to the significance levels, which involve the special function $h(x)$ (e.g. Yao (1993)).

**Power of the test.** In this paragraph we show first on the intuitive level and then more precisely that the proposed test (2.4) has asymptotic power one. That is, under regime $P_{\nu\gamma}$:

$$\max_{1\le m\le n} R_m \;\ge\; \sum_{k=\nu}^{\min(\gamma-1,n)} \exp\left(\sum_{i=k}^{\min(\gamma-1,n)} \ln\left(\frac{g_1(\varepsilon_{1i},\theta_1)}{g_0(Y_i-\beta_{00}-\beta_{01}x_{1i},\theta_0)}\right)\right), \qquad (2.7)$$

where

$$\begin{aligned} E_{\nu\gamma}\ln\left(\frac{g_1(\varepsilon_{1i},\theta_1)}{g_0(Y_i-\beta_{00}-\beta_{01}x_{1i},\theta_0)}\right) &= -E_{\nu\gamma}\ln\left(\frac{g_0(Y_i-\beta_{00}-\beta_{01}x_{1i},\theta_0)}{g_1(\varepsilon_{1i},\theta_1)}\right) \\ &\ge -\ln\left(E_{\nu\gamma}\frac{g_0(Y_i-\beta_{00}-\beta_{01}x_{1i},\theta_0)}{g_1(\varepsilon_{1i},\theta_1)}\right) = 0, \end{aligned}$$

$i \in [\nu, \min(\gamma-1,n)]$. Therefore, under some conditions on $E_{\nu\gamma}\left|\frac{g_1(\varepsilon_{1i},\theta_1)}{g_0(Y_i-\beta_{00}-\beta_{01}x_{1i},\theta_0)}\right|^{-t}$ for some $t$ and $i \in [\nu, \gamma)$, the probability of Type II error of test (2.4) vanishes asymptotically, as $\min(\gamma-1,n)-\nu \to \infty$ and $C$ is fixed.

Now we provide a detailed presentation of the necessary conditions and proof of this fact. Denote

$$a_i \quad = \quad E_{\nu\gamma} \ln \frac{g_1(\varepsilon_{1i}, \theta_1)}{g_0(Y_i - \beta_{00} - \beta_{01} x_{1i}, \theta_0)} \geq 0, \quad i \in [\nu, \min(\gamma - 1, n)]. \qquad (2.8)$$

**Proposition 2.3** *Assume that: for some* $T > 0$*, a sequence* $\{s_i \in (0, \infty), i = \nu, \ldots, \min(\gamma - 1, n)\}$ *and some* $\delta_{\nu\gamma} > 0$*:*

$$e^{ta_i} E_{\nu\gamma} \left| \frac{g_0(Y_i - \beta_{00} - \beta_{01} x_{1i}, \theta_0)}{g_1(\varepsilon_{1i}, \theta_1)} \right|^t \leq e^{\frac{1}{2} s_i t^2} < \infty, \ 0 \leq t \leq T, \ i \in [\nu, \min(\gamma - 1, n)];$$

$$nC < e^{-\delta_{\nu\gamma}} \sum_{k=\nu}^{\min(\gamma-1,n)} e^{\sum_{i=k}^{\min(\gamma-1,n)} a_i}.$$

*Then the probability of type II error*

$$P_{\nu\gamma} \left\{ \max_{1 \leq m \leq n} R_m \leq nC \right\} \quad \leq \quad \begin{cases} e^{-\frac{\delta_{\nu\gamma}^2}{2S_{\nu\gamma}}}, & \text{if } \delta_{\nu\gamma} \leq S_{\nu\gamma} T, \\ e^{-\frac{\delta_{\nu\gamma} T}{2}}, & \text{if } \delta_{\nu\gamma} > S_{\nu\gamma} T, \end{cases}$$

*where* $S_{\nu\gamma} = \sum_{i=\nu}^{\min(\gamma-1,n)} s_i$*.*

**Proof.** By (2.7), we have

$$\begin{aligned}
& P_{\nu\gamma} \left\{ \max_{1 \leq m \leq n} R_m \leq nC \right\} \leq P_{\nu\gamma} \left\{ \sum_{k=\nu}^{w_{\gamma n}} \exp \left( \sum_{i=k}^{w_{\gamma n}} \eta_i \right) \leq nC \right\} \\
& = P_{\nu\gamma} \left\{ \sum_{k=\nu}^{w_{\gamma n}} \exp \left( \sum_{i=k}^{w_{\gamma n}} (a_i + (\eta_i - a_i)) \right) \leq nC, \max_{\nu \leq k \leq w_{\gamma n}} \sum_{i=k}^{w_{\gamma n}} (\eta_i - a_i) \geq -\delta_{\nu\gamma} \right\} \\
& + P_{\nu\gamma} \left\{ \sum_{k=\nu}^{w_{\gamma n}} \exp \left( \sum_{i=k}^{w_{\gamma n}} (a_i + (\eta_i - a_i)) \right) \leq nC, \max_{\nu \leq k \leq w_{\gamma n}} \sum_{i=k}^{w_{\gamma n}} (\eta_i - a_i) < -\delta_{\nu\gamma} \right\} \\
& \leq P_{\nu\gamma} \left\{ \max_{\nu \leq k \leq w_{\gamma n}} \sum_{i=k}^{w_{\gamma n}} (a_i - \eta_i) > \delta_{\nu\gamma} \right\}, \quad \text{where} \\
& w_{\gamma n} = \min(\gamma - 1, n), \ \eta_i = \ln \left( \frac{g_1(\varepsilon_{1i}, \theta_1)}{g_0(Y_i - \beta_{00} - \beta_{01} x_{1i}, \theta_0)} \right).
\end{aligned}$$

Now, the proof of Proposition 2.3 directly follows by a result, which is mentioned in Petrov (1975): Let $\xi_1, \ldots, \xi_n$ be independent random variables, $E\xi_k = 0, \ k = 1, \ldots, n$. If $Ee^{t\xi_k} \leq e^{\frac{d_k}{2} t^2}$, for $0 \leq t \leq T$ and some $d_k > 0, T > 0$ $(k = 1, \ldots, n)$, then

$$P \left\{ \max_{1 \leq m \leq n} \sum_{i=1}^{m} \xi_i \geq x \right\} \quad \leq \quad \begin{cases} e^{-\frac{x^2}{2D}}, & \text{if } 0 \leq x \leq DT, \\ e^{-\frac{xT}{2}}, & \text{if } x > DT, \end{cases}$$

where $D = \sum_{i=1}^{n} d_i$.

**Remark 2.2.** Consider the assumptions in Proposition 2.3 on a simple example. Let for some $0 < a_d \leq a_u < \infty$ and $0 < z < \infty$: $a_d \leq a_i \leq a_u$ and $E_{\nu\gamma} \left| \frac{g_0(Y_i - \beta_{00} - \beta_{01} x_{1i}, \theta_0)}{g_1(\varepsilon_{1i}, \theta_1)} \right|^2 \leq z$ be for all $i \in [\nu, \min(\gamma - 1, n)]$. Hence, we have

$$e^{2a_i} E_{\nu\gamma} \left| \frac{g_0(Y_i - \beta_{00} - \beta_{01} x_{1i}, \theta_0)}{g_1(\varepsilon_{1i}, \theta_1)} \right|^2 \leq e^{\frac{1}{2}s2^2}, \ s \geq |a_u + \frac{1}{2} \ln z|, \ i \in [\nu, \min(\gamma - 1, n)]$$

and the first condition of Proposition 2.3 is satisfied ($T = t = 2$, $s_i = s$ in the formulation of this proposition). Define $\delta_{\nu\gamma} = a_d(\min(\gamma - 1, n) - \nu + 1)^p$, $p \in (1/2, 1)$. Therefore, if

$$nC \ < \ e^{-a_d(\min(\gamma-1,n)-\nu+1)^p} \frac{e^{a_d(\min(\gamma-1,n)-\nu+1)} - 1}{1 - e^{-a_d}}$$

then the second condition of Proposition 2.3 holds, inasmuch as

$$\begin{aligned} e^{-\delta_{\nu\gamma}} \sum_{k=\nu}^{\min(\gamma-1,n)} e^{\sum_{i=k}^{\min(\gamma-1,n)} a_i} &\geq e^{-\delta_{\nu\gamma}} \sum_{k=\nu}^{\min(\gamma-1,n)} e^{(\min(\gamma-1,n)-k+1)a_d} \\ = e^{-a_d(\min(\gamma-1,n)-\nu+1)^p} \frac{e^{a_d(\min(\gamma-1,n)-\nu+1)} - 1}{1 - e^{-a_d}}. \end{aligned}$$

Assume $\delta_{\nu\gamma} \leq S_{\nu\gamma}2 \equiv 2s(\min(\gamma - 1, n) - \nu + 1)$, then Proposition 2.3 shows that the probability of type II error is bounded by an exponentially vanishing term

$$P_{\nu\gamma} \left\{ \max_{1 \leq m \leq n} R_m \leq nC \right\} \ \leq \ e^{-\frac{1}{2s} a_d^2 (\min(\gamma-1,n)-\nu+1)^{2p-1}}.$$

**Remark 2.3.** Note that the proposed test has the same upper bound for the significance level if we consider model (1.1), where $\theta_0$ and $\theta_1$ are multidimensional parameters.

# 3 Problem of Oversight

This section considers the same problem as in Section 2, but $\beta_{10}, \beta_{11}, \beta_{12}, \theta_1$ are unknown. This case has been widely discussed in the literature (e.g. Yakir, Krieger, Pollak (1999) and Lorden, Pollak (2004)) and corresponds to a control problem where the baseline in-control distribution is known and the epidemic state out-of-control distribution is not.

Denote

$$\Lambda_{km}^{(1)} = \prod_{i=k}^{m} \frac{g_1(Y_i - \widehat{\beta}_{10}^{(k,i-1)} - \widehat{\beta}_{11}^{(k,i-1)} x_{1i} - \widehat{\beta}_{12}^{(k,i-1)} x_{2i}, \widehat{\theta}_1^{(k,i-1)})}{g_0(Y_i - \beta_{00} - \beta_{01} x_{1i}, \theta_0)}, \tag{3.9}$$

where $\{\widehat{\beta}_{10}^{(l,m)}, \widehat{\beta}_{11}^{(l,m)}, \widehat{\beta}_{12}^{(l,m)}, \widehat{\theta}_1^{(l,m)}\}$ are some (any) estimators of $\{\beta_{10}, \beta_{11}, \beta_{12}, \theta_1\}$ based upon $\{Y_l, \ldots, Y_m\}$, and $\{\widehat{\beta}_{10}^{(l,l-1)}, \widehat{\beta}_{11}^{(l,l-1)}, \widehat{\beta}_{12}^{(l,l-1)}, \widehat{\theta}_1^{(l,l-1)}\} = \{0, 0, 0, 1\}$. The same idea appears in Robbins and Siegmund (1973) in the context of sequential hypothesis testing and is applied to the change point problem in Dragalin (1997) as well as in Lorden and Pollak (2004). By requiring $\{\widehat{\beta}_{10}^{(k,i-1)}, \widehat{\beta}_{11}^{(k,i-1)}, \widehat{\beta}_{12}^{(k,i-1)}, \widehat{\theta}_1^{(k,i-1)}\}$ not to depend on $Y_i$ one preserves the martingale property of $R_m^{(1)} - m$, where

$$R_m^{(1)} = \sum_{k=1}^{m} \Lambda_{km}^{(1)}, \; m = 1, \ldots, n. \tag{3.10}$$

We propose the following test: reject $H_0$ iff

$$\frac{1}{n} \max_{1 \le m \le n} R_m^{(1)} > C, \tag{3.11}$$

where $R_m^{(1)}$ by (3.10).

**Significance level of the test.**

**Proposition 3.1** *The significance level $\alpha^{(1)}$ of the test satisfies:*

$$\alpha^{(1)} \equiv P_0 \left\{ \max_{1 \le m \le n} R_m^{(1)} > nC \right\} \le 1/C.$$

**Proof.** As mentioned in this section, the sequence $\{R_m^{(1)} - m\}$ is the martingale under $P_0$, hence by applying directly the proof scheme of Proposition 2.1, we conclude the proof of Proposition 3.1

# 4 General Case of the Problem

This section considers the model (1.1) and test hypothesis (1.2), where

$\beta_{00}, \beta_{01}, \theta_0,\ \beta_{10}, \beta_{11}, \beta_{12}, \theta_1$ are unknown. Denote

$$\Lambda_{km}^{(2)} = \prod_{i=k}^{m} \frac{g_1(Y_i - \widehat{\beta}_{10}^{(k,i-1)} - \widehat{\beta}_{11}^{(k,i-1)} x_{1i} - \widehat{\beta}_{12}^{(k,i-1)} x_{2i}, \widehat{\theta}_1^{(k,i-1)})}{g_0(Y_i - \widetilde{\beta}_{00}^{(k,m)} - \widetilde{\beta}_{01}^{(k,m)} x_{1i}, \widetilde{\theta}_0^{(k,m)})}, \tag{4.12}$$

where $\widehat{\beta}_{10}^{(l,m)}, \widehat{\beta}_{11}^{(l,m)}, \widehat{\beta}_{12}^{(l,m)}, \widehat{\theta}_1^{(l,m)}$ are defined in (3.9), $\{\widetilde{\beta}_{00}^{(k,m)}, \widetilde{\beta}_{01}^{(k,m)}, \widetilde{\theta}_0^{(k,m)}\}$ are Maximum Likelihood Estimators (*MLEs*) of $\{\beta_{00}, \beta_{01}, \theta_0\}$ in the $\sigma$-algebra generated by $\{Y_k, \ldots, Y_m\}$:

$$\left\{\widetilde{\beta}_{00}^{(k,m)}, \widetilde{\beta}_{01}^{(k,m)}, \widetilde{\theta}_0^{(k,m)}\right\} \equiv \arg\max_{\beta_{00}^*, \beta_{01}^*, \theta_0^*} \prod_{i=k}^{m} g_0(Y_i - \beta_{00}^* - \beta_{01}^* x_{1i}, \theta_0^*). \tag{4.13}$$

The same idea appears in Gurevich and Vexler (2004) in a different context. By definition (4.13), we obtain the inequality

$$\prod_{i=k}^{m} g_0(Y_i - \widetilde{\beta}_{00}^{(k,m)} - \widetilde{\beta}_{01}^{(k,m)} x_{1i}, \widetilde{\theta}_0^{(k,m)}) \geq \prod_{i=k}^{m} g_0(Y_i - \beta_{00} - \beta_{01} x_{1i}, \theta_0) \tag{4.14}$$

and use it to consider the significant level of a test.

We propose the following test: reject $H_0$ iff

$$\frac{1}{n} \max_{1 \leq m \leq n} R_m^{(2)} > C, \tag{4.15}$$

where $R_m^{(2)} = \sum_{k=1}^{m} \Lambda_{km}^{(2)}$, $\Lambda_{km}^{(2)}$ by (4.12) and the threshold $C > 0$.

**Significance level of the test.**

**Proposition 4.1** *The significance level $\alpha^{(2)}$ of the test satisfies:*

$$\alpha^{(2)} \equiv P_0 \left\{ \max_{1 \leq m \leq n} R_m^{(2)} > nC \right\} \leq 1/C.$$

**Proof.** Following (4.14), we obtain that for all $1 \leq m \leq n$: $R_m^{(2)} \leq R_m^{(1)}$, where $R_m^{(1)}$ by (3.10). Hence, $\alpha^{(2)} \leq \alpha^{(1)}$. Applying Proposition 3.1, we conclude that $\alpha^{(2)} \leq 1/C$.

**Remark 4.1.** Consider the probability of Type II error:

$$P_{\nu\gamma} \left\{ \max_{1 \leq m \leq n} R_m^{(2)} \leq nC \right\} \leq P_{\nu\gamma} \left\{ R_{\min(\gamma-1,n)}^{(2)} \leq nC \right\}.$$

Intuitively, if $\widehat{\beta}_{1m}^{(k,i-1)}, \widehat{\theta}_1^{(k,i-1)},\ m = 0,1,2,\ \nu \le k, i < \gamma$ are *'good'* estimators then for some $q_1 \in (0,1)$:

$$P_{\nu\gamma}\left\{R_{w_{\gamma n}}^{(2)} \le nC\right\} \le P_{\nu\gamma}\left\{\sum_{k=\nu}^{w_{\gamma n}} \Lambda_{k w_{\gamma n}}^{(2)} \le nC\right\}$$
$$\le P_{\nu\gamma}\left\{q_1 \sum_{k=\nu}^{w_{\gamma n}} \exp\left(\sum_{i=k}^{w_{\gamma n}} \ln \frac{g_1(\varepsilon_{1i}, \theta_1)}{g_0(Y_i - \widetilde{\beta}_{00}^{(k,w_{\gamma n})} - \widetilde{\beta}_{01}^{(k,w_{\gamma n})} x_{1i}, \widetilde{\theta}_0^{(k,w_{\gamma n})})}\right) \le nC\right\} + o_{w_{\gamma n}-\nu}(1),$$
$$w_{\gamma n} = \min(\gamma - 1, n),\ o_{w_{\gamma n}-\nu}(1) \to 0 \text{ as } w_{\gamma n} - \nu \to \infty,$$

and, if some conditions are set for the densities $g_0,\ g_1$ and sequences $\{x_{li},\ l = 1,2,\ \nu \le i < \gamma\}$, then for some $\beta_{00}^{'}, \beta_{01}^{'}, \theta_0^{'}$ (such that $\widetilde{\beta}_{0m}^{(k,w_{\gamma n})} \to_{\text{under } H_1} \beta_{00}^{'}$ and $\widetilde{\theta}_0^{(k,w_{\gamma n})} \to_{\text{under } H_1} \theta_0^{'},\ w_{\gamma n} - k \to \infty,\ m = 0,1$ ) and $q_2 \in (0, q_1)$, we have

$$P_{\nu\gamma}\left\{\max_{1 \le m \le n} R_m^{(2)} \le nC\right\}$$
$$\le P_{\nu\gamma}\left\{q_2 \sum_{k=\nu}^{w_{\gamma n}} \exp\left(\sum_{i=k}^{w_{\gamma n}} \ln \frac{g_1(\varepsilon_{1i}, \theta_1)}{g_0((\beta_{10} - \beta_{00}^{'}) + (\beta_{11} - \beta_{01}^{'}) x_{1i} + \beta_{12} x_{2i} + \varepsilon_{1i}, \theta_0^{'})}\right) \le nC\right\}$$
$$+ o_{w_{\gamma n}-\nu}(1).$$

Hence, using the same considerations as in Section 2, we conclude intuitively that the probability of Type II error is asymptotically vanished:

$$P_{\nu\gamma}\left\{\max_{1 \le m \le n} R_m^{(2)} \le nC\right\} \to 0,\ (\min(\gamma - 1, n) - \nu) \to \infty.$$

Note that, if $g_0 = g_1$, $\beta_{01} \ne 0$ and $\beta_{12} = 0$ (or $g_0 = g_1$ and $x_{1i} = b_1 x_{2i} + b_0$ for all $\nu \le i \le \min(\gamma - 1, n)$ and some constants $b_1$ and $b_0$), then by definition of $\{\widetilde{\beta}_{00}^{(k,m)}, \widetilde{\beta}_{01}^{(k,m)}, \widetilde{\theta}_0^{(k,m)}\}$, we have

$$\{\widetilde{\beta}_{0m}^{(\nu,\min(\gamma-1,n))}, \widetilde{\theta}_0^{(\nu,\min(\gamma-1,n))}\} \xrightarrow[\text{under } H_1]{} \{\beta_{0m}^{'} = \beta_{1m}, \theta_0^{'} = \theta_1\},\ m = 0,1,$$

$(\min(\gamma - 1, n) - \nu) \to \infty$ and hence the considerations of this paragraph are misled. Therefore, we consider this case separately in the next section.

# 5 The problem where $\beta_{12}$ is known to be zero

If $\beta_{12} = 0$ (or $g_0 = g_1$ and $x_{1i} = b_1 x_{2i} + b_0$ for some constants $b_1$ and $b_0$) the proposed solution is based on an invariant statistics. Brown, Durbin and Evans (1975), Pollak and

Siegmund (1991) used an invariant statistics in order to remove the influence of unknown parameters under $H_0$ and to preserve a martingale structure. We use the principles of this approach, even though the invariance may not exist for the parameter $\theta_0$.

Let $f_0$ denote the density function under $H_0$ and $f_{km}$ denote the density function under $H_1$ with $\nu = k, \gamma = m$. By the principle of an invariant method we define

$$t_i = Y_i - \frac{x_{11}Y_2 - x_{12}Y_1}{x_{11} - x_{12}} - x_{1i}\frac{Y_1 - Y_2}{x_{11} - x_{12}}, \tag{5.16}$$

under regime $H_0$, $t_i$ does not involve $\beta_{00}, \beta_{01}$.

Applying the rule of average conditional probabilities, we define the following likelihood ratio

$$\begin{aligned}
\Lambda_{km}^{(demo)} &\equiv \frac{f_{km}(t_k, \ldots, t_m)}{f_0(t_k, \ldots, t_m)} = \frac{D_{1km}}{D_{0km}}, \text{ where } k > 2, && (5.17)\\
D_{0km} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_0(t_k, \ldots, t_m | Y_1 = u_1, Y_2 = u_2) dP_0\{Y_1 < u_1, Y_2 < u_2\}\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{i=k}^{m} g_0\left(t_i + \frac{x_{11}u_2 - x_{12}u_1 + (u_1 - u_2)x_{1i}}{x_{11} - x_{12}}, \theta_0\right)\\
&\quad \times g_0(u_1, \theta_0)\, g_0(u_2, \theta_0)\, du_1 du_2,\\
D_{1km} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{km}(t_k, \ldots, t_m \big| Y_1 = u_1, Y_2 = u_2) dP_0\{Y_1 < u_1, Y_2 < u_2\}\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{i=k}^{m}\\
&\quad \times g_1\left(t_i + \rho_0 + \rho_1 x_{1i} + \frac{x_{11}u_2 - x_{12}u_1 + (u_1 - u_2)x_{1i}}{x_{11} - x_{12}}, \theta_1\right)\\
&\quad \times g_0(u_1, \theta_0)\, g_0(u_2, \theta_0)\, du_1 du_2,\\
&\quad \rho_0 \equiv (\beta_{00} - \beta_{10}),\ \ \rho_1 \equiv (\beta_{01} - \beta_{11}).
\end{aligned}$$

Let $\theta_0$, $\theta_1$, $\rho_0$ and $\rho_1$ be unknown. Following the concepts of sections 3, 4 we denote an estimator

$$\Lambda_{km}^{(3)} = \frac{D_{1km}^{(3)}}{D_{0km}^{(3)}(\widetilde{\theta}_0^{(k,m)})} \tag{5.18}$$

of $\Lambda_{km}^{(demo)}$ by (5.17), where

$$\begin{aligned}
D_{0km}^{(3)}(\theta_0^*) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{i=k}^{m} g_0\left(t_i + \frac{x_{11}u_2 - x_{12}u_1 + (u_1-u_2)x_{1i}}{x_{11}-x_{12}}, \theta_0^*\right) \qquad (5.19)\\
&\times g_0\left(u_1, \theta_0^*\right) g_0\left(u_2, \theta_0^*\right) du_1 du_2,\\
D_{1km}^{(3)} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{i=k}^{m} g_1\Big(t_i + \widehat{\rho}_0^{(1,k,i-1)}\\
&+\widehat{\rho}_1^{(1,k,i-1)}x_{1i} + \frac{x_{11}u_2 - x_{12}u_1 + (u_1-u_2)x_{1i}}{x_{11}-x_{12}}, \widehat{\theta}_1^{(k,i-1)}\Big)\\
&\times \prod_{r=1}^{2} g_0\left(u_r, \widehat{\theta'}_0^{(1,k-1)}\right) du_1 du_2,\\
\widetilde{\theta}_0^{(k,n)} &= \arg\max_{\theta_0^*} D_{0kn}^{(3)}(\theta_0^*),
\end{aligned}$$

$\widehat{\theta'}_0^{(1,k-1)}$ is an estimator of $\theta_0$ based upon $Y_1,\ldots,Y_{k-1}$ ($\widehat{\theta'}_0^{(1,0)} = 1$); $\{\widehat{\rho}_0^{(1,k,i-1)}, \widehat{\rho}_1^{(1,k,i-1)}\}$ are some (any) estimators of $\{\rho_0,\rho_1\}$ based upon $Y_1,\ldots,Y_{i-1}$ (for example, $\widehat{\rho}_0^{(1,k,i-1)} = \widehat{\beta}_{00}^{(1,k-1)} - \widehat{\beta}_{10}^{(k,i-1)}$, $\widehat{\rho}_1^{(1,k,i-1)} = \widehat{\beta}_{01}^{(1,k-1)} - \widehat{\beta}_{11}^{(k,i-1)}$); $\widehat{\theta}_1^{(l,m)}$ is an estimator of $\theta_1$ based upon $Y_l,\ldots,Y_m$, and $\{\widehat{\rho}_0^{(1,1,0)}, \widehat{\rho}_1^{(1,1,0)}, \widehat{\theta}_1^{(l,l-1)}\} = \{0,0,0,1\}$.

We propose the following test: reject $H_0$ iff

$$\frac{1}{n}\max_{3\le m\le n} R_m^{(3)} \;>\; C, \qquad (5.20)$$

where $R_m^{(3)} = \sum_{k=3}^{m}\Lambda_{km}^{(3)}$, $\Lambda_{km}^{(3)}$ by (5.18).

**Significance level of the test.**

**Proposition 5.1** *The significance level $\alpha^{(3)}$ of the test satisfies:*

$$\alpha^{(3)} \equiv P_0\left\{\max_{3\le m\le n} R_m^{(3)} > nC\right\} \le 1/C.$$

**Proof.** By (5.17) and the definition of $D_{0km}^{(3)}(\widetilde{\theta}_0^{(k,n)})$ in (5.19), we have

$$\Lambda_{km}^{(3)} \;\le\; Q_{km}^{(3)} := \frac{D_{1km}^{(3)}}{D_{0km}^{(3)}(\theta_0)} = \frac{D_{1km}^{(3)}}{D_{0km}} = \frac{D_{1km}^{(3)}}{f_0(t_k,\ldots,t_m)},$$

therefore

$$\begin{aligned}\alpha^{(3)} &\leq P_0\left\{\max_{3\leq m\leq n}\sum_{k=3}^{m} Q_{km}^{(3)} > nC\right\} \qquad (5.21)\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_0\left\{\max_{3\leq m\leq n}\sum_{k=3}^{m} Q_{km}^{(3)} > nC\Big|Y_1=u_1, Y_2=u_2\right\}\\ &\quad\times dP_0\{Y_1<u_1\}dP_0\{Y_2<u_2\}.\end{aligned}$$

Now, we consider the sequence $\left\{\sum_{k=3}^{m} Q_{km}^{(3)} - m,\ m>3\right\}$, where $Y_1$ and $Y_2$ are fixed (under this condition $t_1,\ldots,t_n$ are independent).

$$\begin{aligned}&E_{H_0,\ Y_1=u_1,Y_2=u_2}\left(Q_{km}^{(3)}\big|(t_1,\ldots,t_{m-1})\right) = \frac{1}{f_0(t_1,\ldots,t_{m-1})} \qquad (5.22)\\ &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{i=k}^{m-1} g_1\Big(t_i+\widehat{\rho}_0^{(1,k,i-1)}\\ &+\widehat{\rho}_1^{(1,k,i-1)}x_{1i}+\frac{x_{11}u_2^*-x_{12}u_1^*+(u_1^*-u_2^*)x_{1i}}{x_{11}-x_{12}},\widehat{\theta}_1^{(k,i-1)}\Big)\prod_{r=1}^{2} g_0\left(u_r^*,\widehat{\theta'}_0^{(1,k-1)}\right)\\ &\times E_{H_0,\ Y_1=u_1,Y_2=u_2}\Big(\frac{1}{f_0(t_m|Y_1=u_1,Y_2=u_2)}\\ &g_1\left(t_m+\widehat{\rho}_0^{(1,k,m-1)}+\widehat{\rho}_1^{(1,k,m-1)}x_{1m}+\frac{x_{11}u_2^*-x_{12}u_1^*+(u_1^*-u_2^*)x_{1m}}{x_{11}-x_{12}},\widehat{\theta}_1^{(k,m-1)}\right)\\ &\Big|(t_1,\ldots,t_{m-1})\Big)du_1^*du_2^* = Q_{k,m-1}^{(3)},\\ &E_{H_0,\ Y_1=u_1,Y_2=u_2}Q_{mm}^{(3)}\Big|(t_1,\ldots,t_{m-1}) = 1.\end{aligned}$$

Therefore, under $\{H_0,\ Y_1=u_1,Y_2=u_2\}$ the sequence

$$\left\{\sum_{k=3}^{m} Q_{km}^{(3)} - m = \sum_{k=3}^{m-1} Q_{km}^{(3)} - (m-1) + (Q_{mm}^{(3)}-1),\ \Im\left(t_1,\ldots,t_m\right),\ m>3\right\}$$

is the martingale. Hence, by applying directly the proof scheme of Proposition 2.1 for $P_0\left\{\max_{3\leq m\leq n}\sum_{k=3}^{m} Q_{km}^{(3)} > nC\Big|Y_1=u_1,Y_2=u_2\right\}$ from (5.21), we have that

$$\begin{aligned}\alpha^{(3)} &\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_0\left\{\max_{3\leq m\leq n}\sum_{k=3}^{m} Q_{km}^{(3)} > nC\Big|Y_1=u_1, Y_2=u_2\right\}\\ &\quad\times dP_0\{Y_1<u_1\}dP_0\{Y_2<u_2\}\\ &\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{C}dP_0\{Y_1<u_1\}dP_0\{Y_2<u_2\}\leq\frac{1}{C}.\end{aligned}$$

This completes the proof of Proposition 5.1.

**Remark 5.1.** We can define the invariant observations of the form

$$t_i^* \;=\; Y_i - \widehat{b}_{00}^{(1,L)} - \widehat{b}_{01}^{(1,L)} x_{1i},$$

where $\left\{\widehat{b}_{00}^{(1,L)}, \widehat{b}_{01}^{(1,L)}\right\} \in \Im(Y_1, \ldots, Y_L)$, $\{Y_1, \ldots, Y_L\}$ is a learning sample (e.g Yakir (1998)) and

$$\begin{aligned}
\widehat{b}_{01}^{(1,L)} &= \frac{\sum_{i=1}^{L} x_{1i} Y_i - \sum_{i=1}^{L} x_{1i} \sum_{i=1}^{L} Y_i \Big/ L}{\sum_{i=1}^{L} x_{1i}^2 - \left(\sum_{i=1}^{L} x_{1i}\right)^2 \Big/ L}, \\
\widehat{b}_{00}^{(1,L)} &= \frac{\sum_{i=1}^{L} Y_i}{L} - \widehat{b}_{01}^{(1,L)} \frac{\sum_{i=1}^{L} x_{1i}}{L}
\end{aligned}$$

are the least squares estimators of $\{\beta_{00}, \beta_{01}\}$. Using the same method as in (5.20), we can obtain a test with a useful upper bound for its level of significance.

# 6 Monte Carlo Simulation Study

**1.** The results reported in the first part of this section focus on the performance of the model

$$\begin{aligned}
Y_i \;&=\; \varepsilon_{0i} I\{i < \nu\} + (\beta x_i + \varepsilon_{1i})\, I\{\nu \leq i < \gamma\} + \varepsilon_{0i} I\{i \geq \gamma\}, \qquad (6.23) \\
&\nu < \gamma,\ \beta = 1,\ \varepsilon_{mi} \sim g_m(u, \theta_m) = \frac{e^{-|u|/\theta_m}}{2\theta_m}, \\
&\theta_m = 1, m = 0, 1,\ i = 1, \ldots, n = 170.
\end{aligned}$$

The samples were generated from the following scheme: for all of these simulations, a sequence of independent variables $\{x_i \sim \text{Uniform}[-1, 1]\}_{i=1}^{n}$ was fixed as the values of the predictor $x$, then 10000 simulations of the random sequence $\{Y_i\}_{i=1}^{n}$ which satisfy (6.23) were executed for each of the results presented in Tables 1 and 2. Figure 1 plots a realization of model (6.23).

**-Fig. 1 (*file: fig1.ps or fig1.bmp*) here-**

Graphical display of the model realization illustrates complexity of decision making on existence of epidemic component in the model, even in this simple case ($\gamma - \nu > 85$). In our application of the following tests:

$$(2.4), \text{ where } R_m = \sum_{k=1}^{m} \prod_{i=k}^{m} \lambda_i(\theta_0, \theta_1, \beta); \tag{6.24}$$

$$(3.11), \text{ where } R_m^{(1)} = \sum_{k=1}^{m} \prod_{i=k}^{m} \lambda_i \left( \theta_0, \widehat{\theta}_1^{(k,i-1)}, \widehat{\beta}^{(k,i-1)} \right); \tag{6.25}$$

$$(4.15), \text{ where } R_m^{(2)} = \sum_{k=1}^{m} \prod_{i=k}^{m} \lambda_i \left( \widetilde{\theta}_0^{(k,m)}, \widehat{\theta}_1^{(k,i-1)}, \widehat{\beta}^{(k,i-1)} \right); \tag{6.26}$$

$$\lambda_i(\theta_0, \theta_1, \beta) = \frac{\theta_0}{\theta_1} e^{-|Y_i - \beta x_i|/\theta_1 + |Y_i|/\theta_0}$$

we used MLEs based on appropriate $\sigma$-algebra. At this rate, $\widehat{\beta}^{(l,l-1)} = 0, \widehat{\theta}_1^{(l,l-1)} = 1$ were applied in tests (6.25), (6.26).

Table 1 displays the simulated significance levels for the proposed tests (6.24)-(6.26) for different thresholds $C$, where $\nu > 170$ in (6.23). From these results, we can experimentally conclude that the upper bounds by Propositions 2.1, 3.1 and 4.1 are reasonable in the considered case.

**-Table 1 here-**

Table 2 presents the simulated power of the tests (6.24)-(6.26) and the simulated power of CUSUM test

$$\max_{1 \le m \le n} \max_{1 \le k \le m} \Lambda_{km} > C^*, \tag{6.27}$$

where $\Lambda_{km} = \prod_{i=k}^{m} \lambda_i(\theta_0, \theta_1, \beta)$ by (6.24), $C^* = 212$ is the critical value (obtained by simulations) corresponding to a significance level 0.05 of this test.

**-Table 2 here-**

Additionally to Table 2, we consider a lower bound for the power of test (6.24) is obtained by Proposition 2.3. By definition (2.8), we have

$$a_i = \frac{1}{2} \int_{-\infty}^{\infty} (|\beta x_i + u| - |u|) e^{-|u|} du = |\beta x_i| + e^{-|\beta x_i|} - 1 > 0, \; i \in [\nu, \gamma). \tag{6.28}$$

Denote, for example, $t = T = 1$, hence

$$e^{a_i} E_{\nu\gamma} \left| \frac{g_0(Y_i - \beta_{00} - \beta_{01} x_{1i}, \theta_0)}{g_1(\varepsilon_{1i}, \theta_1)} \right| = e^{\frac{1}{2} s_i}, \ \ s_i \equiv 2a_i, \quad i \in [\nu, \gamma).$$

Using

$$\delta_{\nu\gamma} < \ln \left( \frac{\sum_{k=\nu}^{\gamma-1} e^{\sum_{i=k}^{\gamma-1} a_i}}{nC} \right)$$

keeps hold of the conditions of Proposition 2.3. In the case, where $\nu = 20$ and $\gamma = 60$, we obtain $S_{\nu\gamma} = \sum_{i=\nu}^{\gamma-1} s_i = 28.7718$, $\sum_{k=\nu}^{\gamma-1} e^{\sum_{i=k}^{\gamma-1} a_i} = 5977578$ and hence $\delta_{\nu\gamma} = 7.47$. That is, Proposition 2.3 shows that the power of the test is greater than 0.621.

**2.** The second part of this section presents 10000-repetition Monte Carlo simulations of the model

$$\begin{aligned} Y_i &= (a + \varepsilon_{0i}) I\{i < \nu\} + (b + \varepsilon_{1i})\, I\{\nu \leq i < \gamma\} + (a + \varepsilon_{0i}) I\{i \geq \gamma\}, \\ & \quad \nu < \gamma, \ a = 1, b = 2, \ \varepsilon_{mi} \sim g_m(u, \theta_m) = \frac{e^{-u/\theta_m}}{\theta_m}, u \geq 0, \\ & \quad \theta_m = 0.75, m = 0, 1, \ i = 1, \ldots, n = 70. \end{aligned} \tag{6.29}$$

We apply the invariant statistics $t_i = Y_i - Y_1$ and the following tests: according to Section 2 approach:

$$\begin{aligned} & \frac{1}{n} \max_{1 \leq m \leq n} R_m > C, \\ & R_m = \sum_{k=1}^{m} \prod_{i=k}^{m} \left[ \begin{array}{ll} \frac{e^{-(Y_i - b)/\theta_1}}{\theta_1}, & \text{if} \ \ Y_i - b \geq 0 \\ 0, & \text{if} \ \ Y_i - b < 0 \end{array} \right] \Big/ \left[ \begin{array}{ll} \frac{e^{-(Y_i - a)/\theta_0}}{\theta_0}, & \text{if} \ \ Y_i - a \geq 0 \\ 0, & \text{if} \ \ Y_i - a < 0 \end{array} \right], \\ & \left( \frac{1}{0} = \infty \right); \end{aligned} \tag{6.30}$$

according to (5.17) approach:

$$\frac{1}{n}\max_{2\le m\le n} R_m^{(demo)} > C, \qquad R_m = \sum_{k=2}^{m}\frac{D_{1km}}{D_{0km}}, \tag{6.31}$$

$$D_{0km} = \frac{1}{\theta_0}\int_0^\infty e^{-u/\theta_0}\prod_{i=k}^{m}\begin{bmatrix} e^{-(t_i+u)/\theta_0}/\theta_0, & \text{if } u \ge -t_i \\ 0, & \text{if } u < -t_i \end{bmatrix} du$$

$$= \frac{1}{\theta_0^{m-k+2}}\int_{A_{0km}}^{\infty} e^{-\left(\sum_{i=k}^m t_i+(m-k+2)u\right)/\theta_0} du$$

$$= \frac{1}{\theta_0^{m-k+1}(m-k+2)} e^{-\left(\sum_{i=k}^m t_i+(m-k+2)A_{0km}\right)/\theta_0}, \; A_{0km} = \max\left(0, \max_{k\le i\le m}(-t_i)\right),$$

$$D_{1km} = \frac{1}{\theta_0}\int_0^\infty e^{-u/\theta_0}\prod_{i=k}^{m}\begin{bmatrix} e^{-(t_i+\rho+u)/\theta_1}/\theta_1, & \text{if } u \ge -t_i-\rho \\ 0, & \text{if } u < -t_i-\rho \end{bmatrix} du$$

$$= \frac{1}{\theta_1^{m-k}(\theta_1+\theta_0(m-k+1))} e^{-\frac{\sum_{i=k}^m (t_i+\rho)}{\theta_1} - A_{1km}\frac{\theta_1+\theta_0(m-k+1)}{\theta_0\theta_1}},$$

$$A_{1km} = \max\left(0, \max_{k\le i\le m}(-t_i-\rho)\right), \rho = a-b;$$

according to (5.18)-(5.20):

$$\frac{1}{n}\max_{2\le m\le n} R_m^{(3)} > C, \qquad R_m^{(3)} = \sum_{k=2}^{m}\frac{D_{1km}^{(3)}}{D_{0km}^{(3)}(\widetilde{\theta}_0^{(k,m)})} \tag{6.32}$$

$$D_{0km}^{(3)}(\widetilde{\theta}_0^{(k,m)}) = \frac{1}{\left(\widetilde{\theta}_0^{(k,m)}\right)^{m-k+1}(m-k+2)} e^{-\left(\sum_{i=k}^m t_i+(m-k+2)A_{0km}\right)/\widetilde{\theta}_0^{(k,m)}},$$

$$D_{1km}^{(3)} = \frac{1}{\widehat{\theta}_0^{(1,k-1)}}\int_0^\infty e^{-u/\widehat{\theta}_0^{(1,k-1)}}$$

$$\prod_{i=k}^{m}\begin{bmatrix} \exp\left(-\frac{(t_i+\widehat{\rho}^{(1,k,i-1)}+u)}{\widehat{\theta}_1^{(k,i-1)}} - \ln\left(\widehat{\theta}_1^{(k,i-1)}\right)\right), & \text{if } u \ge -t_i-\widehat{\rho}^{(1,k,i-1)} \\ 0, & \text{if } u < -t_i-\widehat{\rho}^{(1,k,i-1)} \end{bmatrix} du$$

$$= \frac{1}{1+\widehat{\theta}_0^{(1,k-1)}\sum_{i=k}^m \frac{1}{\widehat{\theta}_1^{(k,i-1)}}}$$

$$\exp\left(-\sum_{i=k}^{m}\left(\frac{(t_i+\widehat{\rho}^{(1,k,i-1)})}{\widehat{\theta}_1^{(k,i-1)}} + \ln\left(\widehat{\theta}_1^{(k,i-1)}\right)\right) - A_{1km}^{(3)}\left(1+\widehat{\theta}_0^{(1,k-1)}\sum_{i=k}^{m}\frac{1}{\widehat{\theta}_1^{(k,i-1)}}\right)\right),$$

$$A_{1km}^{(3)} = \max\left(0, \max_{k\le i\le m}(-t_i-\widehat{\rho}^{(1,k,i-1)})\right),$$

where the MLEs are

$$\begin{aligned}
\widehat{\theta}_0^{(k,m)} &= \left(\sum_{i=k}^{m} t_i + (m-k+2)A_{0km}\right)/(m-k+1),\\
\widehat{\theta}_0^{(1,k-1)} &= \text{if}(k \geq 3)\ \frac{\sum_{i=1}^{k-1}(Y_i - \min_{1\leq j\leq k-1}(Y_j))}{k-1}\ \text{else } 1,\\
\widehat{\theta}_1^{(k,i-1)} &= \text{if}(i \geq k+2)\ \frac{\sum_{l=k}^{i-1}(Y_l - \min_{k\leq j\leq i-1}(Y_j))}{i-k}\ \text{else } 1,\\
\widehat{\rho}^{(1,k,i-1)} &= \widehat{a}^{(1,k-1)} - \widehat{b}^{(k,i-1)} = \min_{1\leq j\leq k-1}(Y_j) - \min_{k\leq j\leq i-1}(Y_j),\\
\widehat{\rho}^{(1,k,k-1)} &= 0.
\end{aligned}$$

The simulated significance levels for the proposed tests (6.30)-(6.32) for different thresholds $C$ and $\nu > 70$ in (6.27) are displayed in the Table 3.

**-Table 3 here-**

Even in the case, where $\nu = 20, \gamma = 26$, the simulated power of the test ((6.30), $C = 20$) was 0.999 (100000 repetitions were specially completed for this conclusion).

Tables 4 gives the simulated powers of the tests (6.31), (6.32) and, for making a comparison, represents the simulated power of CUSUM most powerful test

$$\max_{1\leq m\leq n}\max_{1\leq k\leq m}\left[\left(\frac{\widehat{\theta}_0^{((1,k-1),(m+1,n))}}{\widehat{\theta}_1^{(k,m)}}\right)^{m-k+1} \tag{6.33}\right.$$
$$\prod_{i=k}^{m}\exp\left(-\frac{(Y_i-\widehat{b}^{(k,m)})}{\widehat{\theta}_1^{(k,m)}}+\frac{(Y_i-\widehat{a}^{((1,k-1),(m,n))})I\{Y_i\geq\widehat{a}^{((1,k-1),(m,n))}\}}{\widehat{\theta}_0^{((1,k-1),(m+1,n))}}\right.$$
$$\left.\left.+\frac{(\infty)I\{Y_i<\widehat{a}^{((1,k-1),(m,n))}\}}{\widehat{\theta}_0^{((1,k-1),(m+1,n))}}\right)\right] > C^*,\ \widehat{a}^{((1,k-1),(m,n))} = \min_{i=1,\ldots,k-1,m,\ldots,n} Y_i,$$

where $\widehat{\theta}_0^{((1,k-1),(m+1,n))}$ is the MLE based upon $Y_1,\ldots,Y_{k-1},Y_{m+1},\ldots,Y_n$; $C^* = 96378579$ is the critical value (obtained by simulations) corresponding to the significance level 0.05 of this test (note that the correct CUSUM most powerful test is the test (6.33), where instead of $\widehat{a}^{((1,k-1),(m,n))}$ we should use the estimator $\widehat{a}^{((1,k-1),(m+1,n))}$, however, it causes difficulties in obtaining a simulated significance level for this test).

**-Table 4 here-**

The results of the Monte Carlo simulations experimentally confirm an existence of a practical meaning of the proposed tests and some modifications of them.

**Remark 6.1.** Using some recurrences $\Big($e.g. for test (6.24):

$$R_m = (1 + R_{m-1})\frac{\theta_0}{\theta_1}e^{-|Y_m - \beta x_m|/\theta_1 + |Y_m|/\theta_0}\Big)$$

facilitates programming strategy for the tests considered in this paper. Note that Zeileis et al. (2002) propose some programming methods helpful for practical applications of these tests.

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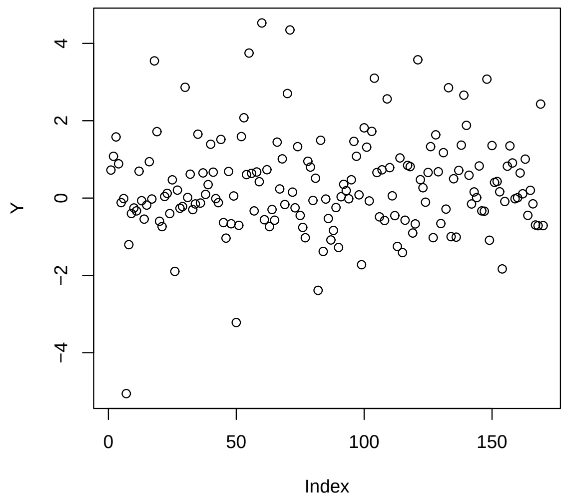


Figure 1: Realization of model (6.23), where $\nu = 85,\ \gamma > 170$

**Table 1:** *The simulated significance levels* $\widehat{\alpha}, \widehat{\alpha}^{(1)}, \widehat{\alpha}^{(2)}$ *of the tests (6.24), (6.25) and (6.26) respectively*

| C | Upper Bound: $1/C$ | $\widehat{\alpha}$ | $\widehat{\alpha}^{(1)}$ | $\widehat{\alpha}^{(2)}$ |
|---|---|---|---|---|
| 10 | 0.1000 | 0.0853 | 0.0737 | 0.0693 |
| 20 | 0.0500 | 0.0442 | 0.0398 | 0.0319 |
| 40 | 0.0250 | 0.0231 | 0.0200 | 0.0190 |
| 50 | 0.0200 | 0.0152 | 0.0142 | 0.0097 |
| 100 | 0.0100 | 0.0081 | 0.0077 | 0.0065 |
| 200 | 0.0050 | 0.0046 | 0.0039 | 0.0035 |
| 300 | 0.0033 | 0.0032 | 0.0029 | 0.0027 |

**Table 2:** *The simulations of the powers* $\widehat{P}^{*}, \widehat{P}, \widehat{P}^{(1)}, \widehat{P}^{(2)}$ *of the tests ((6.27), where* $C^{*} = 212$*) and ((6.24)-(6.26), where* $C = 20$*) respectively*

| $[\nu, \gamma)$ | $[20, 45)$ | $[70, 95)$ | $[125, 150)$ | $[20, 60)$ | $[70, 110)$ | $[20, 70)$ | $[100, 150)$ |
|---|---|---|---|---|---|---|---|
| $\widehat{P}^{*}$ | 0.393 | 0.243 | 0.499 | 0.715 | 0.620 | 0.781 | 0.779 |
| $\widehat{P}$ | 0.351 | 0.272 | 0.478 | 0.722 | 0.609 | 0.782 | 0.808 |
| $\widehat{P}^{(1)}$ | 0.231 | 0.155 | 0.250 | 0.403 | 0.485 | 0.459 | 0.657 |
| $\widehat{P}^{(2)}$ | 0.093 | 0.059 | 0.119 | 0.299 | 0.349 | 0.389 | 0.541 |

**Table 3:** *The simulated significance levels* $\widehat{\alpha}, \widehat{\alpha}^{(demo)}, \widehat{\alpha}^{(3)}$ *of the tests (6.30)-(6.32) respectively*

| C | Upper Bound: $1/C$ | $\widehat{\alpha}$ | $\widehat{\alpha}^{(demo)}$ | $\widehat{\alpha}^{(3)}$ |
|---|---|---|---|---|
| 10 | 0.1000 | 0.0715 | 0.0701 | 0.0581 |
| 20 | 0.0500 | 0.0356 | 0.0381 | 0.0370 |
| 40 | 0.0250 | 0.0190 | 0.0177 | 0.0171 |
| 50 | 0.0200 | 0.0173 | 0.0169 | 0.0168 |
| 100 | 0.0100 | 0.0075 | 0.0079 | 0.0069 |
| 200 | 0.0050 | 0.0041 | 0.0040 | 0.0035 |
| 300 | 0.0033 | 0.0028 | 0.0029 | 0.0027 |

**Table 4:** *The simulations of the powers* $\widehat{P}^{*}, \widehat{P}^{(demo)}, \widehat{P}^{(3)}$ *of the tests ((6.33),* $C^{*} = 96378579$*), ((6.31), (6.32), where* $C = 20$*) respectively*

| $[\nu, \gamma)$ | $[20, 30)$ | $[30, 40)$ | $[50, 60)$ | $[25, 45)$ | $[40, 60)$ |
|---|---|---|---|---|---|
| $\widehat{P}^{*}$ | 0.515 | 0.510 | 0.481 | 0.987 | 0.985 |
| $\widehat{P}^{(demo)}$ | 0.585 | 0.601 | 0.572 | 0.803 | 0.957 |
| $\widehat{P}^{(3)}$ | 0.459 | 0.455 | 0.458 | 0.947 | 0.948 |